Chapter 1

Fourier transforms

1.1 Introduction

Let R be the line parameterized by x. Let f be a complex function on R that is integrable. The Fourier transform $\hat{f} = Ff$ is

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx. \tag{1.1}$$

It is a function on the (dual) real line R' parameterized by k. The goal is to show that f has a representation as an inverse Fourier transform

$$f(x) = \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k)\, \frac{dk}{2\pi}. \tag{1.2}$$

There are two problems. One is to interpret the sense in which these integrals converge. The second is to show that the inversion formula actually holds.

The simplest and most useful theory is in the context of Hilbert space. Let $L^2(R)$ be the space of all (Borel measurable) complex functions such that

$$\|f\|_2^2 = \int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty. \tag{1.3}$$

Then $L^2(R)$ is a Hilbert space with inner product

$$(f, g) = \int \overline{f(x)} g(x)\, dx. \tag{1.4}$$

Let $L^2(R')$ be the space of all (Borel measurable) complex functions such that

$$\|h\|_2^2 = \int_{-\infty}^{\infty} |h(k))|^2\, \frac{dk}{2\pi} < \infty. \tag{1.5}$$

Then $L^2(R')$ is a Hilbert space with inner product

$$(h, u) = \int_{-\infty}^{\infty} \overline{h(k)} u(k)\, \frac{dk}{2\pi}. \tag{1.6}$$

We shall see that the correspondence between f and $\hat{f}$ is a unitary map from $L^2(R)$ onto $L^2(R')$. So this theory is simple and powerful.

1.2 L^1 theory

First, we need to develop the L^1 theory. The space L^1 is a Banach space. Its dual space is L^∞, the space of essentially bounded functions. An example of a function in the dual space is the exponential function $\phi_k(x) = e^{ikx}$. The Fourier transform is then

$$\hat{f}(k) = \langle \phi_k, f \rangle = \int_{-\infty}^{\infty} \overline{\phi_k(x)} f(x)\, dx, \tag{1.7}$$

where ϕ_k is in L^∞ and f is in L^1.

Theorem. If f, g are in $L^1(R)$, then the convolution $f * g$ is another function in $L^1(R)$ defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y) g(y)\, dy. \tag{1.8}$$

Theorem. If f, g are in $L^1(R)$, then the Fourier transform of the convolution is the product of the Fourier transforms:

$$(\widehat{f * g})(k) = \hat{f}(k)\hat{g}(k). \tag{1.9}$$

Theorem. Let $f^*(x) = \overline{f(-x)}$. Then the Fourier transform of f^* is the complex conjugate of $\hat{f}$.

Theorem. If f is in $L^1(R)$, then its Fourier transform $\hat{f}$ is in $L^\infty(R')$ and satisfies $\|\hat{f}\|_\infty \leq \|f\|_1$. Furthermore, $\hat{f}$ is in $C_0(R')$, the space of bounded continuous functions that vanish at infinity.

Theorem. If f is in L^1 and is also continuous and bounded, we have the inversion formula in the form

$$f(x) = \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} e^{ikx} \hat{\delta}_\epsilon(k) \hat{f}(k)\, \frac{dk}{2\pi}, \tag{1.10}$$

where

$$\hat{\delta}_\epsilon(k) = \exp(-\epsilon|k|). \tag{1.11}$$

Proof: The inverse Fourier transform of this is

$$\delta_\epsilon(x) = \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}. \tag{1.12}$$

It is easy to calculate that

$$\int_{-\infty}^{\infty} e^{ikx} \hat{\delta}_\epsilon(k) \hat{f}(k)\, \frac{dk}{2\pi} = (\delta_\epsilon * f)(x). \tag{1.13}$$

However δ_ϵ is an approximate delta function. The result follows by taking $\epsilon \to 0$.

1.3 L^2 theory

The space L^2 is its own dual space, and it is a Hilbert space. It is the setting for the most elegant and simple theory of the Fourier transform.

Lemma. If f is in $L^1(R)$ and in $L^2(R)$, then $\hat{f}$ is in $L^2(R')$, and $\|f\|_2^2 = \|\hat{f}\|_2^2$.

Proof. Let $g = f^* * f$. Then g is in L^1 and is continuous and bounded. Furthermore, the Fourier transform of g is $|\hat{f}(k)|^2$. Thus

$$\|f\|_2^2 = g(0) = \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \hat{\delta}_\epsilon(k) |\hat{f}(k)|^2 \, \frac{dk}{2\pi} = \int_{-\infty}^{\infty} |\hat{f}(k)|^2 \, \frac{dk}{2\pi}. \tag{1.14}$$

Theorem. Let f be in $L^2(R)$. For each a, let $f_a = 1_{[-a,a]}f$. Then f_a is in $L^1(R)$ and in $L^2(R)$, and $f_a \to f$ in $L^2(R)$ as $a \to \infty$. Furthermore, there exists $\hat{f}$ in $L^2(R')$ such that $\hat{f}_a \to \hat{f}$ as $a \to \infty$.

Explicitly, this says that the Fourier transform $\hat{f}(k)$ is characterized by

$$\int_{-\infty}^{\infty} |\hat{f}(k) - \int_{-a}^{a} e^{-ikx} f(x)\, dx|^2 \frac{dk}{2\pi} \to 0 \tag{1.15}$$

as $a \to \infty$.

These arguments show that the Fourier transformation $F : L^2(R) \to L^2(R')$ defined by $Ff = \hat{f}$ is well-defined and preserves norm. It is easy to see from the fact that it preserves norm that it also preserves inner product: $(Ff, Fg) = (f, g)$.

Define the inverse Fourier transform F^* in the same way, so that if h is in $L^1(R')$ and in $L^2(R')$, then F^*h is in $L^2(R)$ and is given by the usual inverse Fourier transform formula. Again we can extend the inverse transformation to $F^* : L^2(R') \to L^2(R)$ so that it preserves norm and inner product.

Now it is easy to check that $(F^*h, f) = (h, Ff)$. Take $h = Fg$. Then $(F^*Fg, f) = (Fg, Ff) = (g, f)$. That is $F^*Fg = g$. Similarly, one may show that $FF^*u = u$. These equations show that F is unitary and that $F^* = F^{-1}$ is the inverse of F. This proves the following result.

Theorem. The Fourier transform F initially defined on $L^1(R) \cap L^2(R)$ extends by continuity to $F : L^2(R) \to L^2(R')$. The inverse Fourier transform F^* initially defined on $L^1(R') \cap L^2(R')$ extends by continuity to $F^* : L^2(R') \to L^2(R)$. These are unitary operators that preserve L^2 norm and preserve inner product. Furthermore, F^* is the inverse of F.

1.4 Absolute convergence

We have seen that the Fourier transform gives a perfect correspondence between $L^2(R)$ and $L^2(R')$. For the other spaces the situation is more complex.

The map from a function to its Fourier transform gives a continuous map from $L^1(R)$ to part of $C_0(R')$. That is, the Fourier transform of an integrable function is continuous and bounded (this is obvious) and approach zero

(Riemann-Lebesgue lemma). Furthermore, this map is one-to-one. That is, the Fourier transform determines the function.

The inverse Fourier transform gives a continuous map from $L^1(R')$ to $C_0(R)$. This is also a one-to-one transformation.

One useful fact is that if f is in $L^1(R)$ and g is in $L^2(R)$, then the convolution $f * g$ is in $L^2(R)$. Furthermore, $\widehat{f * g}(k) = \hat{f}(k)\hat{g}(k)$ is the product of a bounded function with an $L^2(R')$ function, and therefore is in $L^2(R')$.

However the same pattern of the product of a bounded function with an L^2 function can arise in other ways. For instance, consider the translate f_a of a function f in $L^2(R)$ defined by $f_a(x) = f(x-a)$. Then $\widehat{f_a}(k) = \exp(-ika)\hat{f}(k)$. This is also the product of a bounded function with an $L^2(R')$ function.

One can think of this last example as a limiting case of a convolution. Let δ_ϵ be an approximate δ function. Then $(\delta_\epsilon)_a * f$ has Fourier transform $\exp(-ika)\hat{\delta}_\epsilon(k)\hat{f}(k)$. Now let $\epsilon \to 0$. Then $(\delta_\epsilon)_a * f \to f_a$, while $\exp(-ika)\hat{\delta}_\epsilon(k)\hat{f}(k) \to \exp(-ika)\hat{f}(k)$.

Theorem. If f is in $L^2(R)$ and if f' exists (in the sense that f is an integral of f) and if f' is also in $L^2(R)$, then the Fourier transform is in $L^1(R')$. As a consequence f is is $C_0(R)$.

Proof: $\hat{f}(k) = (1/\sqrt{1+k^2}) \cdot \sqrt{1+k^2}\hat{f}(k)$. Since f is in $L^2(R)$, it follows that $\hat{f}(k)$ is in $L^2(R)$. Since f' is in $L^2(R)$, it follows that $k\hat{f}(k)$ is in $L^2(R')$. Hence $\sqrt{1+k^2}\hat{f}(k)$ is in $L^2(R')$. Since $1/\sqrt{1+k^2}$ is also in $L^2(R')$, it follows from the Schwarz inequality that $\hat{f}(k)$ is in $L^1(R')$.

1.5 Fourier transform pairs

There are some famous Fourier transforms.

Fix $\sigma > 0$. Consider the Gaussian

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{x^2}{2\sigma^2}). \tag{1.16}$$

Its Fourier transform is

$$\hat{g}_\sigma(k) = \exp(-\frac{\sigma^2 k^2}{2}). \tag{1.17}$$

Proof: Define the Fourier transform $\hat{g}_\sigma(k)$ by the usual formula. Check that

$$\left(\frac{d}{dk} + \sigma^2 k\right) \hat{g}_\sigma(k) = 0. \tag{1.18}$$

This proves that

$$\hat{g}_\sigma(k) = C \exp(-\frac{\sigma^2 k^2}{2}). \tag{1.19}$$

Now apply the equality of L^2 norms. This implies that $C^2 = 1$. By looking at the case $k = 0$ it becomes obvious that $C = 1$.

Let $\epsilon > 0$. Introduce the Heaviside function $H(k)$ that is 1 for $k > 0$ and 0 for $k < 0$. The two basic Fourier transform pairs are

$$f_\epsilon(x) = \frac{1}{x - i\epsilon} \tag{1.20}$$

with Fourier transform

$$\hat{f}_\epsilon(k) = 2\pi i H(-k) e^{\epsilon k}. \tag{1.21}$$

and its complex conjugate

$$\overline{f_\epsilon(x)} = \frac{1}{x + i\epsilon} \tag{1.22}$$

with Fourier transform

$$\overline{\hat{f}_\epsilon(-k)} = -2\pi i H(k) e^{-\epsilon k}. \tag{1.23}$$

These may be checked by computing the inverse Fourier transform. Notice that f_ϵ and its conjugate are not in $L^1(R)$.

Take $1/\pi$ times the imaginary part. This gives the approximate delta function

$$\delta_\epsilon(x) = \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}. \tag{1.24}$$

with Fourier transform

$$\hat{\delta}_\epsilon(k) = e^{-\epsilon|k|}. \tag{1.25}$$

Take the real part. This gives the approximate principal value of $1/x$ function

$$p_\epsilon(x) = \frac{x}{x^2 + \epsilon^2} \tag{1.26}$$

with Fourier transform

$$\hat{p}_\epsilon(k) = -\pi i [H(k) e^{-\epsilon k} - H(-k) e^{\epsilon k}]. \tag{1.27}$$

1.6 Problems

1. Let $f(x) = 1/(2a)$ for $-a \le x \le a$ and be zero elsewhere. Find the $L^1(R)$, $L^2(R)$, and $L^\infty(R)$ norms of f, and compare them.
2. Find the Fourier transform of f.
3. Find the $L^\infty(R')$, $L^2(R')$, and $L^1(R')$ norms of the Fourier transform, and compare them.
4. Compare the $L^\infty(R')$ and $L^1(R)$ norms for this problem. Compare the $L^\infty(R)$ and $L^1(R')$ norms for this problem.
5. Use the pointwise convergence at $x = 0$ to evaluate an improper integral.
6. Calculate the convolution of f with itself.
7. Find the Fourier transform of the convolution of f with itself. Verify in this case that the Fourier transform of the convolution is the product of the Fourier transforms.

1.7 Poisson summation formula

Theorem: Let f be in $L^1(R)$ with $\hat{f}$ in $L^1(R')$ and such that $\sum_k |\hat{f}(k)| < \infty$. Then

$$2\pi \sum_n f(2\pi n) = \sum_k \hat{f}(k). \tag{1.28}$$

Proof: Let

$$S(t) = \sum_n f(2\pi n + t). \tag{1.29}$$

Since $S(t)$ is 2π periodic, we can expand

$$S(t) = \sum_k a_k e^{ikt}. \tag{1.30}$$

It is easy to compute that

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} S(t) e^{-ikt}\, dt = \frac{1}{2\pi} \hat{f}(k). \tag{1.31}$$

So the Fourier series of $S(t)$ is absolutely summable. In particular

$$S(0) = \sum_k a_k. \tag{1.32}$$