Radial functions and the Fourier transform

Notes for Math 583A, Fall 2008

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1 Area of a sphere

The volume in n dimensions is

$$\mathrm{vol} = d^n\mathbf{x} = dx_1 \cdots dx_n = r^{n-1}\, dr\, d^{n-1}\omega. \tag{1}$$

Here $r = |\mathbf{x}|$ is the radius, and $\omega = \mathbf{x}/r$ it a radial unit vector. Also $d^{n-1}\omega$ denotes the angular integral. For instance, when $n = 2$ it is $d\theta$ for $0 \le \theta \le 2\pi$, while for $n = 3$ it is $\sin(\theta)\, d\theta\, d\phi$ for $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.

The radial component of the volume gives the area of the sphere. The radial directional derivative along the unit vector $\omega = \mathbf{x}/r$ may be denoted

$$\omega d = \frac{1}{r}(x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}) = \frac{\partial}{\partial r}. \tag{2}$$

The corresponding spherical area is

$$\omega d \rfloor \mathrm{vol} = r^{n-1}\, d^{n-1}\omega. \tag{3}$$

Thus when $n = 2$ it is $(1/r)(x\, dy - y\, dx) = r\, d\theta$, while for $n = 3$ it is $(1/r)(x\, dy\, dz + y\, dz\, dx + x\, dx\, dy) = r^2 \sin(\theta)\, d\theta\, d\phi$.

The divergence theorem for the ball B_r of radius r is thus

$$\int_{B_r} \mathrm{div}\ \mathbf{v}\, d^n\mathbf{x} = \int_{S_r} \mathbf{v} \cdot \omega\, r^{n-1} d^{n-1}\omega. \tag{4}$$

Notice that if one takes $\mathbf{v} = \mathbf{x}$, then div $\mathbf{x} = n$, while $\mathbf{x} \cdot \omega = r$. This shows that n times the volume of the ball is r^n times the surface area of the sphere.

Recall that the Gamma function is defined by $\Gamma(z) = \int_0^\infty t^z e^{-t} \frac{dt}{t}$. It is easy to show that $\Gamma(z+1) = z\Gamma(z)$. Since $\Gamma(1) = 1$, it follows that $\Gamma(n) = (n-1)!$. The result $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$ follows reduction to a Gaussian integral. It follows that $\Gamma(\frac{3}{2}) = \frac{1}{2}\pi^{\frac{1}{2}}$.

Theorem 1 *The area of the unit sphere $S_{n-1} \subseteq \mathbf{R}^n$ is*

$$\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}. \tag{5}$$

Thus in 3 dimensions the area of the sphere is $\omega_2 = 4\pi$, while in 2 dimensions the circumference of the circle is $\omega_1 = 2\pi$. In 1 dimension the two points get count $\omega_0 = 2$.

To prove this theorem, consider the Gaussian integral

$$\int_{\mathbf{R}^n} (2\pi)^{-\frac{n}{2}} e^{-\frac{\mathbf{x}^2}{2}} \, d^n\mathbf{x} = 1. \tag{6}$$

In polar coordinates this is

$$\omega_{n-1}(2\pi)^{-\frac{n}{2}} \int_0^\infty e^{-\frac{r^2}{2}} r^{n-1} \, dr = 1. \tag{7}$$

Let $u = r^2/2$. Then this is

$$\omega_{n-1}(2\pi)^{-\frac{n}{2}} 2^{\frac{n-2}{2}} \int_0^\infty e^{-u} u^{\frac{n-2}{2}} \, du = 1. \tag{8}$$

That is

$$\omega_{n-1}\pi^{-\frac{n}{2}} 2^{-1} \Gamma(\frac{n}{2}) = 1. \tag{9}$$

This gives the result.

2 Fourier transform of a power

Theorem 2 *Let $1 < a < n$. The Fourier transform of $1/|x|^a$ is $C_a/|k|^{n-a}$, where*

$$C_a = (2\pi)^{\frac{n}{2}} \frac{2^{\frac{n-a}{2}} \Gamma(\frac{n-a}{2})}{2^{\frac{a}{2}} \Gamma(\frac{a}{2})}. \tag{10}$$

This is not too difficult. It is clear from scaling that the Fourier transform of $1/|x|^a$ is $C/|k|^{n-a}$. It remains to evaluate the constant C.

Take the inner product with the Gaussian. This gives

$$\int_{\mathbf{R}^n} (2\pi)^{-\frac{n}{2}} e^{-\frac{\mathbf{x}^2}{2}} \frac{1}{|x|^a} \, d^n\mathbf{x} = \int_{\mathbf{R}^n} (2\pi)^{-n} e^{-\frac{\mathbf{x}^2}{2}} C \frac{1}{|k|^{n-a}} \, d^n\mathbf{k}. \tag{11}$$

Writing this in polar coordinates gives

$$(2\pi)^{-\frac{n}{2}} \int_0^\infty e^{-\frac{r^2}{2}} r^{n-1-\alpha} \, dr = C(2\pi)^{-n} \int_0^\infty e^{-\frac{r^2}{2}} r^{1-\alpha} \, dr. \tag{12}$$

This in turn gives

$$(2\pi)^{-\frac{n}{2}} 2^{\frac{n-a-2}{2}} \Gamma(\frac{n-a}{2}) = C(2\pi)^{-n} 2^{\frac{a-2}{2}} \Gamma(\frac{a}{2}). \tag{13}$$

3 The Hankel transform

Define the Bessel function

$$J_\nu(t) = \frac{t^\nu}{(2\pi)^{\nu+1}} \omega_{2\nu} \int_0^\pi e^{-it\cos(\theta)} \sin(\theta)^{2\nu} \, d\theta. \tag{14}$$

This makes sense for all real numbers $\nu \geq 0$, but we shall be interested mainly in the cases when ν is an integer or ν is a half-integer. In the case when ν is a half-integer the exponent 2ν is odd, and so it is possible to evaluate the integral in terms of elementary functions. Thus, for example,

$$J_{\frac{1}{2}}(t) = \frac{t^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} 2\pi \int_0^\pi e^{-it\sin(\theta)} \sin(\theta) \, d\theta = \frac{t^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} 2 \frac{\sin(t)}{t}. \tag{15}$$

This is not possible when ν is an integer. Thus for $\nu = 0$ we have the relatively mysterious expression

$$J_0(t) = \frac{1}{\pi} \int_0^\pi e^{it\cos(\theta)} \, d\theta. \tag{16}$$

Fix a value of ν. If we consider a function $g(r)$, its Hankel transform is the function $\hat{g}_\nu(s)$ given by

$$\hat{g}_\nu(s) = \int_0^\infty J_\nu(sr) g(r) r \, dr. \tag{17}$$

We shall see that the Hankel transform is related to the Fourier transform.

4 The radial Fourier transform

The first result is that the radial Fourier transform is given by a Hankel transform. Suppose f is a function on $\mathbf{R}^n$. Its Fourier transform is

$$\hat{f}(\mathbf{k}) = \int e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \, d^n\mathbf{x}. \tag{18}$$

Let $r = |\mathbf{x}|$ and $s = |\mathbf{k}|$. Write $f(\mathbf{x}) = F(r)$ and $\hat{f}(\mathbf{k}) = F_n(s)$.

Theorem 3 *The radial Fourier transform in n dimensions is given in terms of the Hankel transform by*

$$s^{\frac{n-2}{2}} \hat{F}_n(s) = (2\pi)^{\frac{n}{2}} \int_0^\infty J_{\frac{n-2}{2}}(sr) r^{\frac{n-2}{2}} F(r) r \, dr. \tag{19}$$

Here is the proof of the theorem. Introduce polar coordinates with the z axis along $\mathbf{k}$, so that $\mathbf{k} \cdot \mathbf{x} = sr\cos(\theta)$. Suppose that the function is radial, that is, $f(\mathbf{x}) = F(r)$.

$$\hat{f}(\mathbf{k}) = \hat{F}_n(s) = \int_0^\infty \int_0^\pi e^{-isr\cos(\theta)} F(r) \omega_{n-2} \sin(\theta)^{n-2} \, d\theta r^{n-1} \, dr. \tag{20}$$

Use

$$J_{\frac{n-2}{2}}(t) = \frac{t^{\frac{n-2}{2}}}{(2\pi)^{\frac{n}{2}}}\omega_{n-2}\int_0^{\pi} e^{-it\cos(\theta)}\sin(\theta)^{n-2}\,d\theta. \tag{21}$$

For the case $n = 3$ the Bessel function has order 1/2 and has the above expression in terms of elementary functions. So

$$\hat{F}_3(s) = 4\pi\int_0^{\infty}\frac{\sin(sr)}{sr}F(r)r^2\,dr. \tag{22}$$

For $n = 2$ the Bessel function has order 0. We get

$$\hat{F}_2(s) = 2\pi\int_0^{\infty}J_0(sr)F(r)r\,dr. \tag{23}$$