Vector fields and differential forms

William G. Faris

September 25, 2008

Contents

Chapter 1

Forms

1.1 The dual space

The objects that are dual to vectors are 1-forms. A *1-form* is a linear transformation from the n-dimensional vector space V to the real numbers. The 1-forms also form a vector space V^* of dimension n, often called the dual space of the original space V of vectors. If α is a 1-form, then the value of α on a vector $\mathbf{v}$ could be written as $\alpha(\mathbf{v})$, but instead of this we shall mainly use $\alpha \cdot \mathbf{v}$. The condition of being linear says that

$$\alpha \cdot (a\mathbf{u} + b\mathbf{v}) = a\alpha \cdot \mathbf{u} + b\alpha \cdot \mathbf{v}. \tag{1.1}$$

The vector space of all 1-forms is called V^*. Sometimes it is called the *dual space* of V.

It is important to note that the use of the dot in this context is not meant to say that this is the inner product (scalar product) of two vectors. In Part III of this book we shall see how to associate a form $\mathbf{gu}$ to a vector $\mathbf{u}$, and the inner product of $\mathbf{u}$ with $\mathbf{w}$ will then be $\mathbf{gu} \cdot \mathbf{w}$.

There is a useful way to picture vectors and 1-forms. A vector is pictured as an arrow with its tail at the origin of the vector space V. A 1-form is pictured by its contour lines (in two dimensions) or its contour planes (in three dimensions). These are parallel lines or parallel planes that represent when the values of the 1-form are multiples of some fixed small number $\delta > 0$. Sometimes it is helpful to indicate which direction is the direction of increase. The value $\alpha \cdot \mathbf{v}$ of a 1-form α on a vector $\mathbf{v}$ is the value associated with the contour that passes through the head of the arrow.

Each contour line is labelled by a numerical value. In practice one only draws contour lines corresponding to multiples of some fixed small numerical value. Since this numerical value is somewhat arbitrary, it is customary to just draw the contour lines and indicate the direction of increase. The contour line passing through the origin has value zero. A more precise specification of the 1-form would give the numerical value associated with at least one other contour line.

A scalar multiple $c\alpha$ of a 1-form α has contour lines with increased or decreased spacing, and possibly with reversed direction of increase. The sum $\alpha + \beta$ of two 1-forms α, β is defined by adding their values. The sum of two 1-forms may also be indicated graphically by a parallelogram law. The two forms define an array of parallelograms. The contour lines of the sum of the two forms are lines through two (appropriately chosen) corners of the parallelograms.

1.2 Differential 1-forms

A *differential form* is a linear transformation from the vector fields to the reals given by

$$\alpha = \sum_{i=1}^{n} a_i dx_i. \tag{1.2}$$

We identify a vector field $\mathbf{v}$ with the corresponding directional derivative

$$\mathbf{v} = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}. \tag{1.3}$$

The value of α on the vector field $\mathbf{v}$ is

$$\alpha \cdot \mathbf{v} = \sum_{i=1}^{n} a_i v_i. \tag{1.4}$$

If z is a scalar function on M, then it has a *differential* given by

$$dz = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} dx_i. \tag{1.5}$$

This is a special kind of differential form. In general, a differential form that is the differential of a scalar is called an *exact* differential form.

If z is a smooth function on M, and $\mathbf{v}$ is a vector field, then the *directional derivative* of z along $\mathbf{v}$ is

$$dz \cdot \mathbf{v} = \sum_{i=1}^{n} v_i \frac{\partial z}{\partial x_i}. \tag{1.6}$$

It is another smooth function on M.

Theorem 1 (Necessary condition for exactness) *If $\alpha = \sum_{i=1}^{n} a_i dx_i$ is an exact differential form, then its coefficients satisfy the integrability conditions*

$$\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}. \tag{1.7}$$

When the integrability condition is satisfied, then the differential form is said to be *closed.* Thus the theorem says that every exact form is closed.

In two dimensions an exact differential form is of the form

$$dh(x,y) = \frac{\partial h(x,y)}{\partial x}\,dx + \frac{\partial h(x,y)}{\partial y}\,dy. \tag{1.8}$$

If $z = h(x,y)$ this can be written in a shorter notation as

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy. \tag{1.9}$$

It is easy to picture an exact differential form in this two-dimensional case. Just picture contour curves of the function $z = h(x,y)$. These are curves defined by $z = h(x,y) = c$, where the values of c are spaced by some small $\delta > 0$. Notice that adding a constant to z is does not change the differential of z. It also does not change the contour curves of z. For determination of the differential form what is important is not the value of the function, since this has an arbitrary constant. Rather it is the spacing between the contour curves that is essential.

In this picture the exact differential form should be thought of a closeup view, so that on this scale the contour curves look very much like contour lines. So the differential form at a point depends only on the contour lines very near this point.

In two dimensions a general differential form is of the form

$$\alpha = f(x,y)\,dx + g(x,y)\,dy. \tag{1.10}$$

The condition for a closed form is

$$\frac{\partial g(x,y)}{\partial x} = \frac{\partial f(x,y)}{\partial y}. \tag{1.11}$$

If the form is not closed, then it is not exact. The typical differential form is not closed.

We could also write this as

$$\alpha = p\,dx + q\,dy. \tag{1.12}$$

The condition for a closed form is

$$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}. \tag{1.13}$$

It somewhat harder to picture a differential 1-form that is not exact. The idea is to draw contour lines near each point that somehow join to form contour curves. However the problem is that these contour curves now must have end points, in order to keep the density of lines near each point to be consistent with the definition of the differential form.

Example. A typical example of a differential form that is not exact is $y\,dx$. The contour lines are all vertical. They are increasing to the right in the upper half plane, and they are increasing to the left in the lower half plane. However the density of these contour lines must diminish near the x axis, so that some of the lines will have end points at their lower ends (in the upper half plane) or at their upper ends (in the lower half plane).

A differential form may be expressed in various coordinate systems. Say, for instance, that

$$\alpha = p\,dx + q\,dy. \tag{1.14}$$

We may write

$$dx = \frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv, \tag{1.15}$$

$$dy = \frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv. \tag{1.16}$$

Inserting this in the expression for the 1-form α, we obtain

$$\alpha = \left(\frac{\partial x}{\partial u}p + \frac{\partial y}{\partial u}q\right)\,du + \left(\frac{\partial x}{\partial v}p + \frac{\partial y}{\partial v}\right)\,dv. \tag{1.17}$$

1.3 Ordinary differential equations in two dimensions

A classic application of these ideas is ordinary differential equations in the plane. Such an equation is often written in the form

$$p\,dx + q\,dy = 0. \tag{1.18}$$

Here $p = f(x, y)$ and $q = g(x, y)$ are functions of x, y. The equation is determined by the differential form $p\,dx+q\,dy$, but two different forms may determine equivalent equations. For example, if $\mu = h(x, y)$ is a non-zero scalar, then the form $\mu p\,dx + \mu q\,dy$ is a quite different form, but it determines an equivalent differential equation.

If $p\,dx + q\,dy$ is exact, then $p\,dx + q\,dy = dz$, for some scalar z depending on x and y. The solution of the differential equation is then given implicitly by $z = c$, where c is constant of integration.

If $p\,dx + q\,dy$ is not exact, then one looks for an integrating factor μ such that

$$\mu(p\,dx + q\,dy) = dz \tag{1.19}$$

is exact. Once this is done, again the solution of the differential equation is then given implicitly by $z = c$, where c is constant of integration.

Theorem 2 *Suppose that $\alpha = p\,dx + q\,dy$ is a differential form in two dimensions that is non-zero near some point. Then α has a non-zero integrating factor μ near the point, so $\mu\alpha = ds$ for some scalar.*

This theorem follows from the theory of solutions of ordinary differential equations. Finding the integrating factor may not be an easy matter. However, there is a strategy that may be helpful.

Recall that if a differential form is exact, then it is closed. So if μ is an integrating factor, then

$$\frac{\partial \mu p}{\partial y} - \frac{\partial \mu q}{\partial x} = 0. \tag{1.20}$$

This condition may be written in the form

$$p\frac{\partial \mu}{\partial y} - q\frac{\partial \mu}{\partial x} + \left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)\mu = 0. \tag{1.21}$$

Say that by good fortune there is an integrating factor μ that depends only on x. Then this gives a linear ordinary differential equation for μ that may be solved by integration.

Example. Consider the standard problem of solving the linear differential equation

$$\frac{dy}{dx} = -ay + b, \tag{1.22}$$

where a, b are functions of x. Consider the differential form $(ay-b)\,dx+dy$. Look for an integrating factor μ that depends only on x. The differential equation for μ is $-d\mu/dx = a\mu$. This has solution $\mu = e^A$, where A is a function of x with $dA/dx = a$. Thus

$$e^A(ay - b)\,dx + e^A\,dy = d(e^A y - S), \tag{1.23}$$

where S is a function of x with $dS/dx = e^A b$. So the solution of the equation is $y = e^{-A}(S + c)$.

Theorem 3 *Consider a differential form $\alpha = p\,dx + q\,dy$ in two dimensions. Suppose that near some point α is not zero. Suppose also that α is not closed near this point. Then near this point there is a new coordinate system u, v with $\alpha = u\,dv$.*

The proof is to note that if $\alpha = p\,dx + q\,dy$ is not zero, then it has a non-zero integrating factor with $\mu\alpha = dv$. So we can write $\alpha = u\,dv$, where $u = 1/\mu$. Since $u\,dv = p\,dx + q\,dy$, we have $u\partial v/\partial x = p$ and $u\partial v/\partial y = q$. It follows that $\partial q/\partial x - \partial p/\partial y = \partial u/\partial x \partial v/\partial y - \partial u/\partial y \partial v \partial x$. Since this is non-zero, the inverse function theorem shows that this is a legitimate change of coordinates.

The situation is already considerably more complicated in three dimensions, the canonical form is relatively complicated. The differential equations book by Ince [9] treats this situation.

1.4 Problems

1. Exact differentials. Is $(x^2 + y^2)\,dx + 2xy\,dy$ an exact differential form? If so, write it as the differential of a scalar.

2. Exact differentials. Is $(1 + e^x)\,dy + e^x(y - x)\,dx$ an exact differential? If so, write it as the differential of a scalar.

3. Exact differentials. Is $e^y\,dx + x(e^y + 1)\,dy$ an exact differential? If so, write it as the differential of a scalar.

4. Constant differential forms. A differential form usually cannot be transformed into a constant differential form, but there are special circumstances when that can occur. Is it possible to find coordinates u and v near a given point (not the origin) such that

$$-y\,dx + x\,dy = du? \tag{1.24}$$

5. Constant differential forms. A differential form usually cannot be transformed into a constant differential form, but there are special circumstances when that can occur. Is it possible to find coordinates u and v near a given point (not the origin) such that

$$-\frac{y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy = du? \tag{1.25}$$

6. Ordinary differential equations. Solve the differential equation $(xy^2 + y)\,dx - x\,dy = 0$ by finding an integrating factor that depends only on y.

Chapter 2

The exterior derivative

2.1 The exterior product

Let $V \times V$ be the set of ordered pairs $\mathbf{u}, \mathbf{v}$ of vectors in V. A *2-form* σ is an anti-symmetric bilinear transformation $\sigma : V \times V \to \mathbf{R}$. Thus for each fixed $\mathbf{v}$ the function $\mathbf{u} \mapsto \sigma(\mathbf{u}, \mathbf{v})$ is linear, and for each fixed $\mathbf{u}$ the function $\mathbf{v} \mapsto (\mathbf{u}, \mathbf{v})$ is linear. Furthermore, $\sigma(\mathbf{u}, \mathbf{v}) = -\sigma(\mathbf{v}, \mathbf{u})$. The vector space of all 2-forms is denoted $\Lambda^2 V^*$. It is a vector space of dimension $n(n-1)/2$.

A 2-form has a geometric interpretation. First consider the situation in the plane. Given two planar 2-forms, at least one of them is a multiple of the other. So the space of planar 2-forms is one-dimensional. However we should not think of such a 2-form as a number, but rather as a grid of closely spaced points. The idea is that the value of the 2-form is proportional to the number of points inside the parallelogram spanned by the two vectors. The actual way the points are arranged is not important; all that counts is the (relative) density of points. Actually, to specify the 2-form one needs to specify not only the points but also an orientation, which is just a way of saying that the sign of the answer needs to be determined.

In three-dimensional space one can think of parallel lines instead of points. The space of 2-forms in three-dimensional space has dimension 3, because these line can have various directions as well as different spacing. The value of the 2-form on a pair of vectors is proportional to the number of lines passing through the parallelogram spanned by the two vectors. Again, there is an orientation associated with the line, which means that one can perhaps think of each line as a thin coil wound in a certain sense.

The sum of two 2-forms may be given by a geometrical construction that somewhat resembles vector addition.

The *exterior product* (or wedge product) $\alpha \wedge \beta$ of two 1-forms is a 2-form. This is defined by

$$(\alpha \wedge \beta)(\mathbf{u}, \mathbf{v}) = \det \begin{bmatrix} \alpha \cdot \mathbf{u} & \alpha \cdot \mathbf{v} \\ \beta \cdot \mathbf{u} & \beta \cdot \mathbf{v} \end{bmatrix} = (\alpha \cdot \mathbf{u})(\beta \cdot \mathbf{v}) - (\beta \cdot \mathbf{u})(\alpha \cdot \mathbf{v}). \qquad (2.1)$$

Notice that $\alpha \wedge \beta = -\beta \wedge \alpha$. In particular $\alpha \wedge \alpha = 0$.

The exterior product of two 1-forms has a nice geometrical interpretation. On two dimensions each of the two 1-forms is given by a family of parallel lines. The corresponding 2-form consists of the points at the intersection of these lines.

In three dimensions each of the two 1-forms is given by a collection of parallel planes. The corresponding 2-form consists of the lines that are the intersections of these planes.

In a similar way, one can define a 3-form τ as an alternating trilinear function from ordered triples of vectors to the reals. In three dimensions a 3-form is pictured by a density of dots.

One way of getting a 3-form is by taking the exterior product of three 1-forms. The formula for this is

$$(\alpha \wedge \beta \wedge \gamma)(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \det \begin{bmatrix} \alpha \cdot \mathbf{u} & \alpha \cdot \mathbf{v} & \alpha \cdot \mathbf{w} \\ \beta \cdot \mathbf{u} & \beta \cdot \mathbf{v} & \beta \cdot \mathbf{w} \\ \gamma \cdot \mathbf{u} & \gamma \cdot \mathbf{v} & \gamma \cdot \mathbf{w} \end{bmatrix} \tag{2.2}$$

In a similar way one can define r-forms on an n dimensional vector space V. The space of such r-forms is denoted $\Lambda^r V^*$, and it has dimension given by the binomial coefficient $\binom{n}{r}$. It is also possible to take the exterior product of r 1-forms and get an r-form. The formula for this multiple exterior product is again given by a determinant.

The algebra of differential forms is simple. The sum of two r-forms is an r form. The product of an r-form and an s-form is an $r+s$-form. This multiplication satisfies the associative law. It also satisfies the law

$$\beta \wedge \alpha = (-1)^{rs} \alpha \wedge \beta, \tag{2.3}$$

where α is an r-form and β is an s-form. For instance, if $r = s = 1$, then $\alpha \wedge \beta = -\beta \wedge \alpha$. On the other hand, if $r = 1, s = 2$, then $\alpha\beta = \beta\alpha$.

2.2 Differential r-forms

One can also have differential r-forms on a manifold. For instance, on three dimensions one might have a differential 2-form such as

$$\sigma = a\, dy \wedge dz + b\, dz \wedge dx + c\, dx \wedge dy. \tag{2.4}$$

Here x, y, z are arbitrary coordinates, and a, b, c are smooth functions of x, y, z. Similarly, in three dimensions a typical 3-form might have the form

$$\tau = s\, dx \wedge dy \wedge dz. \tag{2.5}$$

Notice that these forms are created as linear combinations of exterior products of 1-forms.

Since these expressions are so common, it is customary in many contexts to omit the explicit symbol for the exterior product. Thus the forms might be written

$$\sigma = a\, dy\, dz + b\, dz\, dx + c\, dx\, dy \tag{2.6}$$

and

$$\tau = s\,dx\,dy\,dz. \tag{2.7}$$

The geometric interpretation of such forms is quite natural. For instance, in the three dimensional situation of these examples, a 1-form is represented by a family of surfaces, possibly ending in curves. Near each point of the manifold the family of surfaces looks like a family of parallel contour planes. A 2-form is represented by a family of curves, possibly ending in points. Near each point of the manifold they look like a family of parallel lines. Similarly, a 3-form is represented by a cloud of points. While the density of points near a given point of the manifold is constant, at distant points of the manifold the densities may differ.

2.3 Properties of the exterior derivative

The exterior derivative of an r-form α is an $r+1$-form $d\alpha$. It is defined by taking the differentials of the coefficients of the r-form. For instance, for the 1-form

$$\alpha = p\,dx + q\,dy + r\,dz \tag{2.8}$$

the differential is

$$d\alpha = dp\,dx + dq\,dy + dr\,dz. \tag{2.9}$$

This can be simplified as follows. First, note that

$$dp = \frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy + \frac{\partial p}{\partial z}\,dz. \tag{2.10}$$

Therefore

$$dp\,dx = \frac{\partial p}{\partial y}\,dy\,dx + \frac{\partial p}{\partial z}\,dz\,dx = -\frac{\partial p}{\partial y}\,dx\,dy + \frac{\partial p}{\partial z}\,dz\,dx. \tag{2.11}$$

Therefore, the final answer is

$$d\alpha = d(p\,dx+q\,dy+r\,dz) = \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right)dy\,dz + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right)dz\,dx + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)dx\,dy. \tag{2.12}$$

Similarly, suppose that we have a 2-form

$$\sigma = a\,dy\,dz + b\,dz\,dx + c\,dx\,dy. \tag{2.13}$$

Then

$$d\sigma = da\,dy\,dz + db\,dz\,dx + dc\,dx\,dy = \frac{\partial a}{\partial x}\,dx\,dy\,dz + \frac{\partial b}{\partial y}\,dy\,dz\,dx + \frac{\partial c}{\partial z}\,dz\,dx\,dy. \tag{2.14}$$

This simplifies to

$$d\sigma = d(a\,dy\,dz + b\,dz\,dx + c\,dx\,dy) = \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right)dx\,dy\,dz. \tag{2.15}$$

The geometrical interpretation of the exterior derivative is natural. Consider first the case of two dimension. If α is a 1-form, then it is given by a family of curves, possibly with end points. The derivative $d\alpha$ corresponds to these end points. They have an orientation depending on which end of the curve they are at.

In three dimensions, if α is a 1-form, then it is given by contour surfaces, possibly ending in curves. The 2-form $d\alpha$ is given by the curves. Also, if σ is a 2-form, then it is given by curves that may terminate. Then $d\sigma$ is a 3-form represented by the termination points.

The exterior derivative satisfies various general properties. The exterior derivative of an r-form is an $r+1$ form. There is a product rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta, \tag{2.16}$$

where α is an r-form and β is an s-form. The reason for the $(-1)^r$ is that the d has to be moved past the r form, and this picks up r factors of -1. Another important property is that applying the exterior derivative twice always gives zero, that is, for an arbitrary s-form β we have

$$dd\beta = 0. \tag{2.17}$$

2.4 The integrability condition

This last property has a geometrical interpretation. Take for example a scalar s. Its differential is $\alpha = ds$, which is an exact differential. Therefore ds is represented by curves without end points (two dimensions) or by surfaces without ending curves (three dimensions). This explains why $d\alpha = dds = 0$.

Similarly, consider a 1-form α in three dimensions. Its differential is a 2-form $\sigma = d\alpha$. The 1-form α is represented by surfaces, which may terminate in closed curves. These closed curves represent the 2 form $d\alpha$. Since they have no end points, we see that $d\sigma = dd\alpha = 0$.

In general, if $d\beta = 0$, then we say that β is a closed form. If $\beta = d\alpha$, we say that β is an exact form. The general fact is that if β is exact, then β is closed. The condition that $d\beta = 0$ is called the integrability condition, since it is necessary for the possibility that β can be integrated to get α with $\beta = d\alpha$.

Example. Consider the 2-form $y\,dx$. This is represented by vertical lines that terminate at points in the plane. The density of these lines is greater as one gets farther from the x axis. The increase is to the right above the x axis, and it is to the left below the y axis. The differential of $y\,dx$ is $dy\,dx = -dx\,dy$. This 2-form represents the cloud of terminating points, which has a uniform density. The usual convention that the positive orientation is counterclockwise. So the orientations of these source points are clockwise. This is consistent with the direction of increase along the contours lines.

2.5 Gradient, curl, divergence

Consider the case of three dimensions. Anyone familiar with vector analysis will notice that if s is a scalar, then the formula for ds resembles the formula for the gradient in cartesian coordinates. Similarly, if α is a 1-form, then the formula for $d\alpha$ resembles the formula for the curl in cartesian coordinates. The formula $dds = 0$ then corresponds to the formula curl grad $s = 0$.

In a similar way, if σ is a 2-form, then the formula for $d\sigma$ resembles the formula for the divergence in cartesian coordinates. The formula $dd\alpha = 0$ then corresponds to the formula div curl $\mathbf{v} = 0$.

There are, however, important distinctions. First, the differential form formulas take the same form in arbitrary coordinate systems. This is not true for the formulas for the divergence, curl, and divergence. The reason is that the usual definitions of divergence, curl, and divergence are as operations on vector fields, not on differential forms. This leads to a much more complicated theory, except for the very special case of cartesian coordinates on Euclidean space. We shall examine this issue in detail in the third part of this book.

Second, the differential form formulas have natural formulations for manifolds of arbitrary dimension. While the gradient and divergence may also be formulated in arbitrary dimensions, the curl only works in three dimensions.

This does not mean that notions such as gradient of a scalar (a vector field) or divergence of a vector field (a scalar) are not useful and important. Indeed, in some situations they play an essential role. However one should recognize that these are relatively complicated objects. Their nature will be explored in the second part of this book (for the divergence) and in the third part of this book (for the gradient and curl).

The same considerations apply to the purely algebraic operations, at least in three dimensions. The exterior product of two 1-forms resembles in some way the cross product of vectors, while the exterior product of a 1-form and a 2-form resembles a scalar product of vectors. Thus the wedge product of three 1-forms resembles the triple scalar product of vector analysis. Again these are not quite the same thing, and the relation will be explored in the third part of this book.

2.6 Problems

1. Say that the differential 1-form $\alpha = p\,dx + q\,dy + r\,dz$ has an integrating factor $\mu \neq 0$ such that $\mu\alpha = ds$. Prove that $\alpha \wedge d\alpha = 0$. Also, express this condition as a condition on p, q, r and their partial derivatives.

2. Show that $\alpha = dz - y\,dx - dy$ has no integrating factor.

3. Show that the differential 1-form $\alpha = yz\,dx + xz\,dy + dz$ passes the test for an integrating factor.

4. In the previous problem it might be difficult to guess the integrating factor. Show that $\mu = e^{xy}$ is an integrating factor, and find s with $\mu\alpha = ds$.

5. The differential 2-form $\omega = (2xy - x^2)\,dx\,dy$ is of the form $\omega = d\alpha$, where α is a 1-form. Find such an α. Hint: This is too easy; there are many solutions.

6. The differential 3-form $\sigma = (yz + x^2z^2 + 3xy^2z)\,dx\,dy\,dz$ is of the form $\sigma = d\omega$, where ω is a 2-form. Find such an ω. Hint: Many solutions.

7. Let $\sigma = xy^2z\,dy\,dz - y^3z\,dz\,dx + (x^2y + y^2z^2)\,dx\,dy$. Show that this 2-form σ satisfies $d\sigma = 0$.

8. The previous problem gives hope that $\sigma = d\alpha$ for some 1-form α. Find such an α. Hint: This may require some experimentation. Try α of the form $\alpha = p\,dx + q\,dy$, where p, q are functions of x, y, z. With luck, this may work. Remember that when integrating with respect to z the constant of integration is allowed to depend on x, y.

Chapter 3

Integration and Stokes's theorem

3.1 One-dimensional integrals

A one-dimensional manifold C is described by a single coordinate t. Consider an interval on the manifold bounded by $t = a$ and $t = b$. There are two possible orientations of this manifold, from $t = a$ to $t = b$, or from $t = b$ to $t = a$. Suppose for the sake of definiteness that the manifold has the first orientation. Then the differential form $f(t)\,dt$ has the integral

$$\int_C f(t)\,dt = \int_{t=a}^{t=b} f(t)\,dt. \tag{3.1}$$

If s is another coordinate, then t is related to s by $t = g(s)$. Furthermore, there are numbers p, q such that $a = g(p)$ and $b = g(q)$. The differential form is thus $f(t)\,dt = f(g(s))g'(s)\,ds$. The end points of the manifold are $s = p$ and $s = q$. Thus

$$\int_C f(t)\,dt = \int_{s=p}^{s=q} f(g(s))g'(s)\,ds. \tag{3.2}$$

The value of the integral thus does not depend on which coordinate is used.

Notice that this calculation depends on the fact that $dt/ds = g'(s)$ is nonzero. However we could also consider a smooth function u on the manifold that is not a coordinate. Several points on the manifold could give the same value of u, and du/ds could be zero at various places. However we can express $u = h(s)$ and $du/ds = h'(s)$ and define an integral

$$\int_C f(u)\,du = \int_{s=p}^{s=q} f(h(s))h'(s)\,ds. \tag{3.3}$$

Thus the differential form $f(u)\,du$ also has a well-defined integral on the manifold, even though u is not a coordinate.

3.2 Integration on manifolds

Next look at the two dimensional case. Say that we have a coordinate system x, y in a two-dimensional oriented manifold. Consider a region R bounded by curves $x = a$, $x = b$, and by $y = c$, $y = d$. Suppose that the orientation is such that one goes around the region in the order a, b then c, d then b, a then d, c. Then the differential form $f(x, y)\, dx\, dy$ has integral

$$\int_R f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right],\ dx. \tag{3.4}$$

The limits are taken by going around the region in the order given by the orientation, first a, b then c, d. We could also have taken first b, a then d, c and obtained the same result.

Notice, by the way, that we could also define an integral with $dy\, dx$ in place of $dx\, dy$. This would be

$$\int_R f(x,y)\,dy\,dx = \int_b^a \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_b^a f(x,y)\,dx\right],\ dy. \tag{3.5}$$

The limits are taken by going around the region in the order given by the orientation, first c, d then b, a. We could also have taken d, c then a, b and obtained the same result. This result is precisely the negative of the previous result. This is consistent with the fact that $dy\, dx = -dx\, dy$.

These formula have generalizations. Say that the region is given by letting x go from a to b and y from $h(x)$ to $k(x)$. Alternatively, it might be given by letting y go from c to d and x from $p(y)$ to $q(y)$. This is a more general region than a rectangle, but the same kind of formula applies:

$$\int_R f(x,y)\,dx\,dy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)\,dx\right] dy = \int_a^b \left[\int_{h(x)}^{k(x)} f(x,y)\,dy\right],\ dx. \tag{3.6}$$

There is yet one more generalization, to the case where the differential form is $f(u, v)\, du\, dv$, but u, v do not form a coordinate system. Thus, for instance, the 1-form du might be a multiple of dv at a certain point, so that $du\, dv$ would be zero at that point. However we can define the integral by using the customary change of variable formula:

$$\int_R f(u,v)\,du\,dv = \int_R f(u,v)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right) dx\,dy. \tag{3.7}$$

In fact, since $du = \partial u/\partial x\, dx + \partial u/\partial y\, dy$ and $dv = \partial v/\partial x\, dx + \partial v/\partial y\, dy$, this is just saying that the same differential form has the same integral.

In fact, we could interpret this integral directly as a limit of sums involving only the u, v increments. Partition the manifold by curves of constant x and constant y. This divides the manifold into small regions that look something like parallelograms. Then we could write this sum as

$$\int_R f(u,v)\,du\,dv \approx \sum f(u,v)\left(\Delta u_x\,\Delta v_y - \Delta v_x\,\Delta u_y\right). \tag{3.8}$$

Here the sum is over the parallelograms. The quantity Δu_x is the increment in u from x to $x+\Delta x$, keeping y fixed, along one side of the parallelogram. The quantity Δv_y is the increment in v from y to $y+\Delta y$, keeping y fixed, along one side of the parallelogram. The other quantities are defined similarly. The u, v value is evaluated somewhere inside the parallelogram. The minus sign seems a bit surprising, until one realizes that going around the oriented boundary of the parallelogram the proper orientation makes a change from x to $x+\Delta x$ followed by a change from y to $y+\Delta y$, or a change from y to $y+\Delta y$ followed by a change from $x+\Delta x$ to x. So both terms have the form $\Delta u\,\Delta v$, where the changes are now taken along two sides in the proper orientation, first the change in u, then the change in v.

3.3 The fundamental theorem

The fundamental theorem of calculus says that for every scalar function s we have

$$\int_C ds = s(Q) - s(P). \tag{3.9}$$

Here C is an oriented path from point P to point Q. Notice that the result does not depend on the choice of path. This is because ds is an exact form.

As an example, we can take a path in space. Then $ds = \partial s/\partial x\,dx + \partial s/\partial y\,dy + \partial s/\partial z\,dz$. So

$$\int_C ds = \int_C \frac{\partial s}{\partial x}\,dx + \frac{\partial s}{\partial y}\,dy + \frac{\partial z}{\partial z}\,dz = \int_C \left(\frac{\partial s}{\partial x}\frac{dx}{dt} + \frac{\partial s}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial z}\frac{dz}{dt}\right) dt. \tag{3.10}$$

By the chain rule this is just

$$\int_C ds = \int_C \frac{ds}{dt}\,dt = s(Q) - s(P). \tag{3.11}$$

3.4 Green's theorem

The next integral theorem is Green's theorem. It says that

$$\int_R \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy = \int_{\partial R} p\,dx + q\,dy. \tag{3.12}$$

Here R is an oriented region in two dimensional space, and ∂R is the curve that is its oriented boundary. Notice that this theorem may be stated in the succinct form

$$\int_R d\alpha = \int_{\partial R} \alpha. \tag{3.13}$$

The proof of Green's theorem just amounts to applying the fundamental theorem of calculus to each term. Thus for the second term one applies the fundamental theorem of calculus in the x variable for fixed y.

$$\int_R \frac{\partial q}{\partial x}\,dx\,dy = \int_c^d \left[\int_{C_y} q\,dx\right]\,dy = \int_c^d \left[q(C_y^+) - q(C_y^-)\right]\,dy. \tag{3.14}$$

This is

$$\int_c^d q(C_y^+)\,dy + \int_d^c q(C_y^-)\,dy = \int_{\partial R} q\,dy. \tag{3.15}$$

The other term is handled similarly, except that the fundamental theorem of calculus is applied with respect to the x variable for fixed y. Then such regions can be pieced together to give the general Green's theorem.

3.5 Stokes's theorem

The most common version of Stokes's theorem says that for a oriented two dimensional surface S in a three dimensional manifold with oriented boundary curve ∂S we have

$$\int_S \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right)\,dy\,dz + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right)\,dz\,dx + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\,dx\,dy = \int_{\partial S} (p\,dx + q\,dy + r\,dz). \tag{3.16}$$

Again this has the simple form

$$\int_S d\alpha = \int_{\partial S} \alpha. \tag{3.17}$$

This theorem reduces to Green's theorem. The idea is to take coordinates u, v on the surface S and apply Green's theorem in the u, v coordinates. In the theorem the left hand side is obtained by taking the form $p\,dx + q\,dy + r\,dz$ and applying d to it. The key observation is that when the result of this is expressed in the u, v coordinates, it is the same as if the form $p\,dx + q\,dy + r\,dz$ were first expressed in the u, v coordinates and then d were applied to it. In this latter form Green's theorem applies directly.

Here is the calculation. To make it simple, consider only the $p\,dx$ term. Then taking d gives

$$d(p\,dx) = \left(\frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy + \frac{\partial p}{\partial z}\,dz\right)\,dx = \frac{\partial p}{\partial z}\,dz\,dx - \frac{\partial p}{\partial y}\,dx\,dy. \tag{3.18}$$

In u, v coordinates this is

$$d(p\,dx) = \left[\frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right) - \frac{\partial p}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\right]\,du\,dv. \tag{3.19}$$

There are four terms in all.

Now we do it in the other order. In u, v coordinates we have

$$p\,dx = p\frac{\partial x}{\partial u}\,du + p\frac{\partial x}{\partial v}\,dv. \tag{3.20}$$

Taking d of this gives

$$d\left(p\frac{\partial x}{\partial u}\,du + p\frac{\partial x}{\partial v}\,dv\right) = \left[\frac{\partial}{\partial u}\left(p\frac{\partial x}{\partial v}\right) - \frac{\partial}{\partial v}\left(p\frac{\partial x}{\partial u}\right)\right]\,du\,dv. \tag{3.21}$$

The miracle is that the second partial derivatives cancel. So in this version

$$d\left(p\frac{\partial x}{\partial u}\,du + p\frac{\partial x}{\partial v}\,dv\right) = \left[\frac{\partial p}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial p}{\partial v}\frac{\partial x}{\partial u}\right]\,du\,dv. \tag{3.22}$$

Now we can express $\partial p/\partial u$ and $\partial p/\partial v$ by the chain rule. This gives at total of six terms. But two of them cancel, so we get the same result as before.

3.6 Gauss's theorem

Let W be an oriented three dimensional region, and let ∂W be the oriented surface that forms its boundary. Then Gauss's theorem states that

$$\int_W \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right)\,dx\,dy\,dz = \int_{\partial W} a\,dy\,dz + b\,dz\,dx + c\,dx\,dy. \tag{3.23}$$

Again this has the form

$$\int_W d\sigma = \int_{\partial W} \sigma, \tag{3.24}$$

where now σ is a 2-form. The proof of Gauss's theorem is similar to the proof of Green's theorem.

3.7 The generalized Stokes's theorem

The generalized Stoke's theorem says that

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega. \tag{3.25}$$

Here ω is a $(k-1)$-form, and $d\omega$ is a k-form. Furthermore, Ω is a k dimensional region, and $\partial\Omega$ is its $(k-1)$-dimensional oriented boundary. The forms may be expressed in arbitrary coordinate systems.

3.8 References

A classic short but rigorous account of differential forms is given in the book of Spivak [15]. The book by Agricola and Friedrich [1] gives a more advanced treatment. Other books on differential forms include those by Cartan [2], do Carmo [3], Edelen [4], Flanders [7], Screiber [14], and Weintraub [17]. There are also advanced calculus texts by Edwards [5] and by Hubbard and Hubbard [8].

There are many sources for tensor analysis; a classical treatment may be found in Lovelock and Rund [10]. There is a particularly unusual and sophisticated treatment in the book of Nelson [12]. Differential forms are seen to be special kinds of tensors: covariant alternating tensors.

The most amazing reference that this author has encountered is an elementary book by Weinreich [16]. He presents the geometric theory of differential forms in pictures, and these pictures capture the geometrical essence of the situation. The principal results of the theory are true by inspection. However his terminology is most unusual. He treats only the case of dimension three. Thus he has the usual notion of covariant 1-form, 2-form, and 3-form. In his terminology the corresponding names for these are stack, sheaf, and scalar density (or swarm). There are also corresponding contravariant objects corresponding to what are typically called 1-vector, 2-vector (surface element), and 3-vector (volume element). The names in this case are arrow, thumbtack, and scalar capacity. The correspondence between his objects and the usual tensors may actually be slightly more complicated than this, but the intent is certainly to explicate the usual calculus geometrically. In particular, he gives geometric explanations of the usual algebraic and differential operations in all these various cases.

3.9 Problems

1. Let C be the curve $x^2 + y^2 = 1$ in the first quadrant from $(1, 0)$ to $(0, 1)$. Evaluate
$$\int_C xy\,dx + (x^2 + y^2)\,dy. \tag{3.26}$$

2. Let C be a curve from $(2, 0)$ to $(0, 3)$. Evaluate
$$\int_C 2xy\,dx + (x^2 + y^2)\,dy. \tag{3.27}$$

3. Consider the problem of integrating the differential form
$$\alpha = -\frac{y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy \tag{3.28}$$
from $(1, 0)$ to $(-1, 0)$ along some curve avoiding the origin. There is are an infinite set of possible answers, depending on the curve. Describe all such answers.

4. Let R be the region $x^2 + y^2 \leq 1$ with $x \geq 0$ and $y \geq 0$. Let ∂R be its boundary (oriented counterclockwise). Evaluate directly
$$\int_{\partial R} xy\,dx + (x^2 + y^2)\,dy. \tag{3.29}$$

5. This continues the previous problem. Verify Green's theorem in this special case, by explicitly calculating the appropriate integral over the region R.

6. Let
$$\alpha = -y\,dx + x\,dy + xy\,dz. \tag{3.30}$$
Fix $a > 0$. Consider the surface S that is the hemisphere $x^2 + y^2 + z^2 = a^2$ with $z \geq 0$. Integrate α over the boundary ∂S of this surface (a counterclockwise circle in the x, y plane).

7. This continues the previous problem. Verify Stokes's theorem in this special case, by explicitly calculating the appropriate integral over the surface S.

8. Let $\sigma = xy^2z\,dy\,dz - y^3z\,dz\,dx + (x^2y + y^2z^2)\,dx\,dy$. Integrate σ over the sphere $x^2 + y^2 + z^2 = a^2$. Hint: This should be effortless.

Chapter 4

The divergence theorem

4.1 Contraction

There is another operation called *interior product* (or contraction). In the case of interest to us, it is a way of defining the product of a vector with a k-form to get a $k-1$ form. We shall mainly be interested in the case when $k = 1, 2, 3$. When $k = 1$ this is already familiar. For a 1-form α the interior product $\mathbf{u}\rfloor\alpha$ is defined to be the scalar $\alpha \cdot \mathbf{v}$.

The interior product of a vector $\mathbf{u}$ with a 2-form σ is a 1-form $\mathbf{u}\rfloor\sigma$. It is defined by

$$(\mathbf{u}\rfloor\sigma) \cdot \mathbf{v} = \sigma(\mathbf{u}, \mathbf{v}). \tag{4.1}$$

This has a nice picture in two dimensions. The vector $\mathbf{u}$ is an arrow. In two dimensions the 2-form σ is given by a density of points. The contour lines of the interior product 1-form are parallel to the arrow. The get them, arrange the points defining the 2-form to be spaced according to the separation determined by the arrow (which may require some modification in the other direction to preserve the density). Then take the contour lines to be spaced according to the new arrangement of the points. These contour lines are the contour lines corresponding to the interior product 1-form.

In three dimensions the 2-form σ is given by lines. The arrow $\mathbf{u}$ and the lines determining σ determine a family of parallel planes. To get these contour planes, do the following. Arrange the lines that determine σ to be spaced according to the separation determined by the arrow (which may require some modification in the other direction to preserve the density). Then take the contour planes to be spaced according to the new separation between the lines. The resulting planes are the contour planes of the interior product 1-form.

The interior product $\mathbf{u}\rfloor\omega$ of a vector $\mathbf{u}$ with a 3-form ω is a 2-form $\mathbf{u}\rfloor\omega$. It is defined by

$$(\mathbf{u}\rfloor\omega)(\mathbf{v}, \mathbf{w}) = \omega(\mathbf{u}, \mathbf{v}, \mathbf{w}). \tag{4.2}$$

(The case of a general r-form is similar.)

The picture is similar. Consider three dimensions. The vector $\mathbf{u}$ is an arrow, and the associated 2-form $\mathbf{u}\rfloor\omega$ is given by lines that are parallel to the arrow. To get these contour lines, do the following. Arrange the points that determine ω to be spaced according to the separation determined by the arrow. Then take the contour lines to be spaced according to the new separation between the points.

One interesting property of the interior product is that if α is an r-form and β is an s-form, then

$$\mathbf{u}\rfloor(\alpha \wedge \beta) = (\mathbf{u}\rfloor\alpha) \wedge \beta + (-1)^r \alpha \wedge (\mathbf{u}\rfloor\beta). \tag{4.3}$$

This is a kind of triple product identity.

In particular, we may apply this when $r = 1$ and $s = n$. Since β is an n-form, it follows that $\alpha \wedge \beta = 0$. Hence we have in this special case

$$(\alpha \cdot \mathbf{u})\beta = \alpha \wedge (\mathbf{u}\rfloor\beta). \tag{4.4}$$

Another application is with two 1-forms β and γ. In this case it gives

$$\mathbf{a}\rfloor(\beta \wedge \gamma) = (\beta \cdot \mathbf{a})\gamma - (\gamma \cdot \mathbf{a})\beta. \tag{4.5}$$

So the interior product of a vector with $\beta \wedge \gamma$ is a linear combination of β and γ.

Later we shall see the connection with classical vector algebra in three dimensions. The exterior product $\beta \wedge \gamma$ is an analog of the cross product, while $\alpha \wedge \beta \wedge \gamma$ is an analog of the triple scalar product. The combination $-\mathbf{a}\rfloor(\beta \wedge \gamma)$ will turn out to be an analog of the triple vector product.

4.2 Duality

Consider an n-dimensional manifold. The new feature is a given n-form, taken to be never zero. We denote this form by vol. In coordinates it is of the form

$$\mathrm{vol} = \sqrt{g}\, du_1 \cdots du_n. \tag{4.6}$$

This coefficient $\sqrt{g}$ depends on the coordinate system. The choice of the notation $\sqrt{g}$ for the coefficient will be explained in the following chapter. (Then $\sqrt{g}$ will be the square root of the determinant of the matrix associated with the Riemannian metric for this coordinate system.)

The most common examples of volume forms are the volume in $\mathrm{vol} = dx\, dy\, dz$ in cartesian coordinates and the same volume $\mathrm{vol} = r^2 \sin(\theta)\, dr\, d\theta\, d\phi$ in spherical polar coordinates. The convention we are using for spherical polar coordinates is that θ is the co-latitude measured from the north pole, while ϕ is the longitude. We see from these coordinates that the $\sqrt{g}$ factor for cartesian coordinates is 1, while the $\sqrt{g}$ factor for spherical polar coordinates is $r^2 \sin(\theta)$.

In two dimensions it is perhaps more natural to call this area. So in cartesian coordinates area $= dx\, dy$, while in polar coordinates area $= r\, dr\, d\phi$.

For each scalar field s there is an associated n-form $s\,\mathrm{vol}$. The scalar field and the n-form determine each other in an obvious way. They are said to be dual to each other, in a certain special sense.

For each vector field $\mathbf{v}$ there is an associated $n-1$ form given by $\mathbf{v}\rfloor\mathrm{vol}$. The vector field and the $n-1$ form are again considered to be dual to each other, in this same sense. If $\mathbf{v}$ is a vector field, then $\mathbf{v}\rfloor\mathrm{vol}$ might be called the corresponding *flux*. It is an $n-1$ form that describes how much $\mathbf{v}$ is penetrating a given $n-1$ dimensional surface.

In two dimensions a vector field is of the form

$$\mathbf{u} = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}. \tag{4.7}$$

The area form is

$$\mathrm{area} = \sqrt{g}\,du\,dv. \tag{4.8}$$

The corresponding flux is

$$\mathbf{u}\rfloor\mathrm{area} = \sqrt{g}(a\,dv - b\,du). \tag{4.9}$$

In three dimensions a vector field is of the form

$$\mathbf{u} = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} + c\frac{\partial}{\partial w}. \tag{4.10}$$

The volume form is

$$\mathrm{vol} = \sqrt{g}\,du\,dv\,dw. \tag{4.11}$$

The corresponding flux is

$$\sqrt{g}(a\,dv\,dw + b\,dw\,du + c\,du\,dv). \tag{4.12}$$

4.3 The divergence theorem

The divergence of a vector field $\mathbf{v}$ is defined to be the scalar $\mathrm{div}\ \mathbf{v}$ such that

$$d(\mathbf{u}\rfloor\mathrm{vol}) = \mathrm{div}\ \mathbf{u}\,\mathrm{vol}. \tag{4.13}$$

In other words, it is the dual of the differential of the dual.

The general divergence theorem then takes the form

$$\int_W \mathrm{div}\ \mathbf{u}\,\mathrm{vol} = \int_{\partial W} \mathbf{u}\rfloor\mathrm{vol}. \tag{4.14}$$

In two dimensions the divergence theorem says that

$$\int_R \frac{1}{\sqrt{g}}\left(\frac{\partial\sqrt{g}a}{\partial u} + \frac{\partial\sqrt{g}b}{\partial v}\right)\mathrm{area} = \int_{\partial R}\sqrt{g}(a\,dv - b\,du). \tag{4.15}$$

Here the area form is $\sqrt{g}\,du\,dv$, where the particular form of $\sqrt{g}$ is that associated with the u, v coordinate system. Notice that the coefficients in the vector field are expressed with respect to a coordinate basis. We shall see in the next part of this book that this is not the only possible choice.

A marvellous application of the divergence theorem in two dimensions is the formula

$$\int_R dx\, dy = \frac{1}{2}\int_{\partial R} x\, dy - y\, dx. \tag{4.16}$$

This says that one can determine the area by walking around the boundary. It is perhaps less mysterious when one realizes that $x\, dy - y\, dx = r^2\, d\phi$.

In three dimensions the divergence theorem says that

$$\int_W \frac{1}{\sqrt{g}}\left(\frac{\partial\sqrt{g}a}{\partial u} + \frac{\partial\sqrt{g}b}{\partial v} + \frac{\partial\sqrt{g}c}{\partial w}\right) \mathrm{vol} = \int_{\partial W} \sqrt{g}(a\, dv\, dw + b\, dw\, du + c\, du\, dv). \tag{4.17}$$

Here the volume form is $\sqrt{g}\, du\, dv\, dw$, where the particular form of $\sqrt{g}$ is that associated with the u, v, w coordinate system. Again the coefficients a, b, c of the vector field are expressed in terms of the coordinate basis vectors $\partial/\partial u, \partial/\partial v, \partial/\partial w$. This is the the only possible kind of basis for a vector field, so in some treatments the formulas will appear differently. They will be ultimately equivalent in terms of their geometrical meaning.

The divergence theorem says that the integral of the divergence of a vector field over W with respect to the volume is the integral of the flux of the vector field across the bounding surface ∂W. A famous application in physics is when the vector field represents the electric field, and the divergence represents the density of charge. So the amount of charge in the region determines the flux of the electric field through the boundary.

4.4 Integration by parts

An important identity for differential forms is

$$d(s\omega) = ds \wedge \omega + s d\omega. \tag{4.18}$$

This gives an integration by parts formula

$$\int_W ds \wedge \omega + \int_W s d\omega = \int_{\partial W} s\omega. \tag{4.19}$$

Apply this to $\omega = \mathbf{u}\rfloor\mathrm{vol}$ and use $ds \wedge \mathbf{u}\rfloor\mathrm{vol} = ds \cdot \mathbf{u}\,\mathrm{vol}$. This gives the divergence identity

$$\mathrm{div}\ (s\mathbf{u}) = ds \cdot \mathbf{u} + s\mathrm{div}\ \mathbf{u}. \tag{4.20}$$

From this we get another important integration by parts identity

$$\int_W ds \cdot \mathbf{u}\,\mathrm{vol} + \int_W s\mathrm{div}\ \mathbf{u}\,\mathrm{vol} = \int_{\partial W} s\mathbf{u}\rfloor\mathrm{vol}. \tag{4.21}$$

4.5 Problems

1. Let $r^2 = x^2 + y^2 + z^2$, and let

$$\mathbf{v} = \frac{1}{r^3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right). \tag{4.22}$$

   Let $\mathrm{vol} = dx\,dy\,dz$. Show that

$$\sigma = \mathbf{v}\rfloor\mathrm{vol} = \frac{1}{r^3}(x\,dy\,dz + y\,dz\,dx + z\,dx\,dy). \tag{4.23}$$

2. In the preceding problem, show directly that $d\sigma = 0$ away from $r = 0$.

3. Find σ in spherical polar coordinates. Hint: This can be done by blind computation, but there is a better way. Express $\mathbf{v}$ in spherical polar coordinates, using Euler's theorem

$$r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}. \tag{4.24}$$

   Then use $\mathrm{vol} = r^2\sin(\theta)\,dr\,d\theta\,d\phi$ to calculate $\sigma = \mathbf{v}\rfloor\mathrm{vol}$.

4. In the preceding problem, show that $d\sigma = 0$ away from $r = 0$ by a spherical polar coordinate calculation.

5. Let S be the sphere of radius $a > 0$ centered at the origin. Calculate the integral of σ over S.

6. Let Q be the six-sided cube with side lengths $2L$ centered at the origin. Calculate the integral of σ over Q. Hint: Given the result of the preceding problem, this should be effortless.

Chapter 5

The metric

5.1 Inner product

An *inner product* on a vector space V is a real function g that takes a pair of input vectors $\mathbf{u}, \mathbf{v}$ and produces a number $\mathsf{g}\mathbf{u} \cdot \mathbf{v}$. It is required to be a bilinear, symmetric, positive, non-degenerate form. That is, it satisfies the following axioms:

1. The form is bilinear: The function $\mathsf{g}\mathbf{u} \cdot \mathbf{v}$ is linear in $\mathbf{u}$ and also linear in $\mathbf{v}$.

2. The form is symmetric: $\mathsf{g}\mathbf{u} \cdot \mathbf{v} = \mathsf{g}\mathbf{v} \cdot \mathbf{u}$.

3. The form is non-degenerate: $\mathsf{g}\mathbf{u} \cdot \mathbf{u} = 0$ implies $\mathbf{u} = 0$.

4. The form is positive: $\mathsf{g}\mathbf{u} \cdot \mathbf{u} \geq 0$,

An inner product g defines a linear transformation $\mathsf{g} : V \to V^*$. That is, the value of g on $\mathbf{u}$ in V is the linear function from V to the real numbers that sends $\mathbf{v}$ to $\mathsf{g}\mathbf{u} \cdot \mathbf{v}$. Thus $\mathsf{g}\mathbf{u}$ is such a function, that is, an element of the dual space V^*.

Since the form g is non-degenerate, the linear transformation g from V to V^* is an isomorphism of vector spaces. Therefore it has an inverse $\mathsf{g}^{-1} : V^* \to V$. Thus if ω is a linear form in V^*, the corresponding vector $\mathbf{u} = \mathsf{g}^{-1}\omega$ is the unique vector $\mathbf{u}$ such that $\mathsf{g}\mathbf{u} \cdot \mathbf{v} = \omega \cdot \mathbf{v}$.

In short, once one has a given inner product, one has a tool that tends to erase the distinction between a vector space and its dual space. It is worth noting that in relativity theory there is a generalization of the notion of inner product in which the form is not required to be positive. However it still gives such an isomorphism between vector space and dual space.

5.2 Riemannian metric

A smooth assignment of an inner product for the tangent vectors at each point of a manifold is called a *Riemannian metric*. It is very convenient to choose coordinates so that the Riemannian metric is diagonal with respect to this coordinate system. In this case it has the form

$$\mathsf{g} = h_1^2\,du_1^2 + h_2^2\,du_2^2 + \cdots + h_n^2\,du_n^2. \tag{5.1}$$

Here each coefficient h_i is a function of the coordinates $u_1, \ldots, u_n$. The differentials is not interpreted in the sense of differential forms. Rather, what this means is that g takes vector fields to differential forms by

$$\mathsf{g}\left(a_1\frac{\partial}{\partial u_1} + \cdots a_n\frac{\partial}{\partial u_n}\right) = a_1 h_1^2\,du_1 + \cdots a_n h_n^2\,du_n \tag{5.2}$$

It is not always possible to find such a coordinate system for which the Riemannian metric is diagonal. However this can always be done when the dimension $n \leq 3$, and it is very convenient to do so. Such a coordinate system is called a system of *orthogonal coordinates*. See the book by Eisenhart [6] for a discussion of this point.

When we have orthogonal coordinates, it is tempting to make the basis vectors have length one. Thus instead of using the usual coordinate basis vectors $\frac{\partial}{\partial u_i}$ one uses the normalized basis vectors $\frac{1}{h_i}\frac{\partial}{\partial u_i}$. Similarly, instead of using the usual coordinate differential forms du_i one uses the normalized differentials $h_i\,du_i$. Then

$$\mathsf{g}\left(a_1\frac{1}{h_1}\frac{\partial}{\partial u_1} + \cdots a_n\frac{1}{h_n}\frac{\partial}{\partial u_n}\right) = a_1 h_1\,du_1 + \cdots a_n h_n\,du_n \tag{5.3}$$

When you use the normalized basis vectors, the coefficients do not change.

In orthogonal coordinates the volume is given by

$$\mathrm{vol} = \sqrt{g}\,du_1 \cdots du_n = h_1 \cdots h_n du_1 \wedge \cdots \wedge du_n. \tag{5.4}$$

A simple example of orthogonal coordinates is that of polar coordinates r, ϕ in the plane. These are related to cartesian coordinates x, y by

$$x = r\cos(\phi) \tag{5.5}$$

$$y = r\sin(\phi) \tag{5.6}$$

The Riemannian metric is expressed as

$$\mathsf{g} = dr^2 + r^2\,d\phi^2. \tag{5.7}$$

The normalized basis vectors are $\frac{\partial}{\partial r}$ and $\frac{1}{r}\frac{\partial}{\partial \phi}$. The normalized basis forms are dr and $r\,d\phi$. The area form is $r\,dr \wedge d\phi$.

Warning: Even though coordinate forms like $d\phi$ are closed forms, a normalized form like $r\,d\phi$ need not be a closed form. In fact, in this particular case $d(r\phi) = dr \wedge d\phi \neq 0$.

Another example of orthogonal coordinates is that of spherical polar coordinates r, θ, ϕ. These are related to cartesian coordinates x, y, z by

$$x = r\cos(\phi)\sin(\theta) \tag{5.8}$$

$$y = r\sin(\phi)\sin(\theta) \tag{5.9}$$

$$z = r\cos(\theta) \tag{5.10}$$

The Riemannian metric is expressed as

$$\mathsf{g} = dr^2 + r^2\,d\theta^2 + r^2\sin^2(\theta)\,d\phi^2. \tag{5.11}$$

The normalized basis vectors are $\frac{\partial}{\partial r}$ and $\frac{1}{r}\frac{\partial}{\partial \theta}$ and $\frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi}$. The normalized basis forms are dr and $r\,d\theta$ and $r\sin(\theta)\,d\phi$. The volume form is $r^2\sin(\theta)\,dr \wedge d\theta \wedge d\phi$.

In these examples one could always use cartesian coordinates. However there are manifolds that cannot be naturally described by cartesian coordinates, but for which orthogonal coordinates are available. A simple example is the sphere of constant radius a. The Riemannian metric is expressed by

$$\mathsf{g} = a^2\,d\theta^2 + a^2\sin^2(\theta)\,d\phi^2. \tag{5.12}$$

The normalized basis vectors are $\frac{1}{a}\frac{\partial}{\partial \theta}$ and $\frac{1}{a\sin(\theta)}\frac{\partial}{\partial \phi}$. The normalized basis forms are $a\,d\theta$ and $a\sin(\theta)\,d\phi$. The area form is $a^2\sin(\theta)\,d\theta \wedge d\phi$.

5.3 Gradient and divergence

If f is a scalar field, then its *gradient* is

$$\nabla f = \text{grad } f = \mathsf{g}^{-1}df. \tag{5.13}$$

Since du is a 1-form, and the inverse of the metric g^{-1} maps 1-forms to vector fields, the gradient ∇f is a vector field.

In orthogonal coordinates ∇f has the form

$$\nabla f = \sum_{i=1}^{n} \frac{1}{h_i^2}\frac{\partial f}{\partial u_i}\frac{\partial}{\partial u_i}. \tag{5.14}$$

In terms of normalized basis vectors this has the equivalent form

$$\nabla f = \sum_{i=1}^{n} \frac{1}{h_i}\frac{\partial f}{\partial u_i}\frac{1}{h_i}\frac{\partial}{\partial u_i}. \tag{5.15}$$

If $\mathbf{u}$ is a vector field, then its divergence $\nabla \cdot \mathbf{u}$ is a scalar field given by requiring that

$$(\text{div } \mathbf{u})\text{vol} = (\nabla \cdot \mathbf{u})\text{vol} = d(\mathbf{u}\rfloor\text{vol}). \tag{5.16}$$

Here vol $= h_1 \cdots h_n \, du_1 \wedge \cdots du_n$ is the volume form. Say that $\mathbf{u}$ has an expression in terms of normalized basis vectors of the form

$$\mathbf{u} = \sum_{i=1}^{n} a_i \frac{1}{h_i} \frac{\partial}{\partial u_i}. \tag{5.17}$$

Then

$$\operatorname{div} \mathbf{u} = \nabla \cdot \mathbf{u} = \sum_{i=1}^{n} \frac{1}{h_1 \cdots h_n} \frac{\partial}{\partial u_i} \left(\frac{h_1 \cdots h_n}{h_i} a_i \right). \tag{5.18}$$

5.4 Gradient dynamics

A scalar function f has both a differential df and a gradient $\mathtt{g}^{-1} df$. What can the gradient do that the differential cannot do? Well, the gradient is a vector field, so it has an associated system of differential equations

$$\frac{du_i}{dt} = \frac{1}{h_i^2} \frac{\partial f}{\partial u_i}. \tag{5.19}$$

Along a solution of this equation the function f satisfies

$$\frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} \frac{\partial u_i}{\partial t} = \sum_{i=1}^{n} \frac{1}{h_i^2} \left(\frac{\partial f}{\partial u_i} \right)^2 \geq 0. \tag{5.20}$$

In more geometrical language this says that

$$\frac{df}{dt} = df \cdot \mathtt{g}^{-1} df \geq 0. \tag{5.21}$$

Along every solution f is increasing in time. If instead you want decrease, you can follow the negative of the gradient.

5.5 The Laplace operator

The Laplace operator ∇^2 is defined as

$$\nabla^2 f = \nabla \cdot \nabla f. \tag{5.22}$$

This can also be written

$$\nabla^2 f = \operatorname{div} \operatorname{grad} f. \tag{5.23}$$

In coordinates the Laplacian has the form

$$\nabla^2 f = \frac{1}{h_1 \cdots h_n} \sum_{i=1}^{n} \frac{\partial}{\partial u_i} \left(\frac{h_1 \cdots h_n}{h_i^2} \frac{\partial f}{\partial u_i} \right) \tag{5.24}$$

For example, in three dimensions with cartesian coordinates it is

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{5.25}$$

In spherical polar coordinates it is

$$\nabla^2 f = \frac{1}{r^2 \sin(\theta)} \left[\frac{\partial}{\partial r} r^2 \sin(\theta) \frac{\partial f}{\partial r} + \frac{\partial}{\partial \theta} \sin(\theta) \frac{\partial f}{\partial \theta} + \frac{\partial}{\partial \phi} \frac{1}{\sin(\theta)} \frac{\partial f}{\partial \phi} \right]. \tag{5.26}$$

This is often written

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial f}{\partial r} + \frac{1}{r^2} \left[\frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \sin(\theta) \frac{\partial f}{\partial \theta} + \frac{1}{\sin^2(\theta)} \frac{\partial^2 f}{\partial \phi^2} \right]. \tag{5.27}$$

5.6 Curl

The remaining objects are in three dimensions.

The cross product of two vectors $\mathbf{v}$ and $\mathbf{w}$ is defined as the unique vector $\mathbf{v} \times \mathbf{w}$ such that

$$(\mathbf{v} \times \mathbf{w}) \rfloor \mathrm{vol} = \mathbf{g}\mathbf{v} \wedge \mathbf{g}\mathbf{w}. \tag{5.28}$$

In other words, it is the operation on vectors that corresponds to the exterior product on forms.

The curl of a vector field $\mathbf{v}$ is defined by

$$(\mathrm{curl}\ \mathbf{v}) \rfloor \mathrm{vol} = d(\mathbf{g}\mathbf{v}). \tag{5.29}$$

The curl has a rather complicated coordinate representation. Say that in a system of orthogonal coordinates

$$\mathbf{v} = a \frac{1}{h_u} \frac{\partial}{\partial u} + b \frac{1}{h_v} \frac{\partial}{\partial v} + c \frac{1}{h_w} \frac{\partial}{\partial w}. \tag{5.30}$$

Thus the vector field is expressed in terms of normalized basis vectors. Then

$$\mathbf{g}\mathbf{v} = a h_u\, du + b h_v\, dv + c h_w\, dw. \tag{5.31}$$

So

$$d(\mathbf{g}\mathbf{v}) = \left(\frac{\partial h_w c}{\partial v} - \frac{\partial h_v b}{\partial w} \right) dv \wedge dw + \left(\frac{\partial h_u a}{\partial w} - \frac{\partial h_w c}{\partial u} \right) dw \wedge du + \left(\frac{\partial h_v b}{\partial u} - \frac{\partial h_u a}{\partial v} \right) du \wedge dv. \tag{5.32}$$

It follows that

$$\mathrm{curl}\ \mathbf{v} = \frac{1}{h_v h_w} \left(\frac{\partial h_w c}{\partial v} - \frac{\partial h_v b}{\partial w} \right) \frac{1}{h_u} \frac{\partial}{\partial u} + \frac{1}{h_u h_w} \left(\frac{\partial h_u a}{\partial w} - \frac{\partial h_w c}{\partial u} \right) \frac{1}{h_v} \frac{\partial}{\partial h_v} + \frac{1}{h_u h_v} \left(\frac{\partial h_v b}{\partial u} - \frac{\partial h_u a}{\partial v} \right) \frac{1}{h_w} \frac{\partial}{\partial h_w}. \tag{5.33}$$

The reason for writing it this way is to express it again in terms of normalized basis vectors. Notice also that if we express the derivatives as normalized derivatives, then the expression is reasonably natural. For instance, the first term is $1/h_w$ times the derivative $(1/h_v)\partial/\partial v)$ of h_w times the coefficient. The only odd thing is that the h_w is inside the derivative, while the $1/h_w$ is outside the derivative.

It is easy to see that curl grad $f = 0$ and that div curl $\mathbf{v} = 0$.

Stokes's theorem says that

$$\int_S \text{curl } \mathbf{v} \rfloor \text{vol} = \int_{\partial S} \mathbf{g}\mathbf{v}. \tag{5.34}$$

Of course, this is just saying that

$$\int_S d(\mathbf{g}\mathbf{v}) = \int_{\partial S} \mathbf{g}\mathbf{v}, \tag{5.35}$$

which is much simpler, since most of the effect of the metric has now cancelled out.

5.7 Problems

1. This problem is three dimensional. Compute the Laplacian of $1/r$ via a cartesian coordinate calculation.

2. This problem is three dimensional. Compute the Laplacian of $1/r$ via spherical polar coordinates.

Chapter 6

Length and area

6.1 Length

Sometimes it is useful to consider coordinates on a manifold that are not orthogonal coordinates. The simplest case is that of a two-dimensional manifold. Write the metric as

$$\mathsf{g} = E\,du^2 + 2F\,du\,dv + G\,dv^2. \tag{6.1}$$

Here E, F, G are functions of u, v. They of course depend on the choice of coordinates. What is required is that $E > 0, G > 0$ and the determinant $EF - G^2 > 0$. When $F = 0$ we are in the case of orthogonal coordinates.

One way that such a metric arises is from a surface in three-dimensional space. Suppose the metric is given in orthogonal coordinates x, y, z by $h_x^2 dx^2 + h_y^2 dy^2 + h_z^2 dz^2$. (If one chooses cartesian coordinates, then $h_x = h_y = h_z = 1$.) The length of a curve is

$$s = \int_C \sqrt{h_x^2\,dx^2 + h_y^2\,dy^2 + h_z^2\,dz^2}. \tag{6.2}$$

The meaning of this equation is that

$$s = \int_a^b \sqrt{h_x^2\left(\frac{dx}{dt}\right)^2 + h_y^2\left(\frac{dy}{dt}\right)^2 + h_z^2\left(\frac{dz}{dt}\right)^2}\,dt, \tag{6.3}$$

where t is a coordinate on the curve, and the end points are where $t = a$ and $t = b$.

Suppose that the curve is in the surface. Then the length is

$$s = \int_a^b \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\,dt. \tag{6.4}$$

Here the coefficients are

$$E = h_x^2\left(\frac{\partial x}{\partial u}\right)^2 + h_y^2\left(\frac{\partial y}{\partial u}\right)^2 + h_z^2\left(\frac{\partial z}{\partial u}\right)^2, \tag{6.5}$$

and

$$F = h_x^2 \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + h_y^2 \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + h_z^2 \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}, \tag{6.6}$$

and

$$G = h_x^2 \left(\frac{\partial x}{\partial v}\right)^2 + h_y^2 \left(\frac{\partial y}{\partial v}\right)^2 + h_z^2 \left(\frac{\partial z}{\partial v}\right)^2. \tag{6.7}$$

Of course for cartesian coordinates x, y, z we have $h_x = h_y = h_z = 1$. This gives the explicit formula for the metric on the surface in terms of the equations giving x, y, z in terms of u, v that define the surface. Often one writes the result for the length of a curve in the surface in the form

$$s = \int_C \sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}. \tag{6.8}$$

This just means that one can use any convenient parameter.

6.2 Area

The formula for the area of a surface is

$$A = \int_S \text{area} = \int_S \sqrt{g}\,du \wedge dv = \int_S \sqrt{EG - F^2}\,du \wedge dv. \tag{6.9}$$

Here $g = EG - F^2$ is the determinant of the metric tensor. This is particularly simple in the case of orthogonal coordinates, in which case $F = 0$.

As an example, take the surface given in cartesian coordinates by $z = x^2 + y^2$ with $z \leq 1$. There are various parameterizations. Take, for instance, $x = r\cos(\phi)$, $y = r\sin(\phi)$, $z = r^2$. Then $E = 1 + 4r^2$, $F = 0$, and $G = r^2$. So with these parameters the area form is $\sqrt{g}\,dr \wedge d\phi = r\sqrt{1 + 4r^2}\,dr \wedge d\phi$. The area is 2π times the integral from 0 to 1 of $r\sqrt{1 + 4r^2}\,dr$. The area is thus $(\pi/6)(5^{\frac{3}{2}} - 1)$.

Again, take the surface $z = x^2 + y^2$ with $z \leq 1$. This time use x, y as parameters. Then $E = 1 + 4x^2$, $F = 4xy$, and $G = 1 + 4y^2$. So with these parameters the area form is $\sqrt{g}\,dx \wedge dy = \sqrt{1 + 4x^2 + 4y^2}\,dx \wedge dy$. This is integrated over the region $x^2 + y^2 = 1$. The answer is of course the same.

There is an alternative expression for the surface area of a surface inside Euclidean space that is sometimes convenient. This is

$$A = \int_S \text{area} = \int_S \sqrt{h_y^2 h_z^2 \left(\frac{dy \wedge dz}{du \wedge dv}\right)^2 + h_z^2 h_x^2 \left(\frac{dz \wedge dx}{du \wedge dv}\right)^2 + h_x^2 h_y^2 \left(\frac{dx \wedge dy}{du \wedge dv}\right)^2}\, du \wedge dv. \tag{6.10}$$

Again for cartesian coordinates x, y, z we have $h_x = h_y = h_z = 1$. Here a fraction such as $dy \wedge dz$ divided by $du \wedge dv$ is a ratio of 2-forms on the surface S. As we know, such a ratio is just a Jacobian determinant of y, z with respect to u, v.

Once again, take the surface given in cartesian coordinates by $z = x^2 + y^2$ with $z \leq 1$. Use x, y as parameters. Then $(dy \wedge dz)/(dx \wedge dy) = -2x$ and $(dz \wedge dx)/(dx \wedge dy) = -2y$ and $(dx \wedge dy)/(dx \wedge dy) = 1$. So the area form is $\sqrt{g}\,dx \wedge dy = \sqrt{4x^2 + 4y^2 + 1}\,dx \wedge dy$ as before.

6.3 Divergence and Stokes theorems

This section has a discussion of the form that the divergence and Stokes's theorems have in the context of orthogonal coordinates.

The first topic is the divergence theorem in two dimensions. Say that the vector field $\mathbf{v}$ has an expression in terms of normalized basis vectors of the form

$$\mathbf{v} = a\frac{1}{h_u}\frac{\partial}{\partial u} + b\frac{1}{h_v}\frac{\partial}{\partial v}. \tag{6.11}$$

Recall that the area form is

$$\mathrm{area} = h_u h_v \, du \, dv. \tag{6.12}$$

Then the corresponding differential 2-form is

$$\mathbf{v}\rfloor\mathrm{area} = a h_v \, dv - b h_u \, du. \tag{6.13}$$

The divergence theorem in two dimensions is obtained by applying Greens's theorem for 1-forms to this particular 1-form. The result is

$$\int_R \left[\frac{1}{h_u h_v}\left(\frac{\partial}{\partial u}(h_v a) + \frac{\partial}{\partial v}(h_u b)\right)\right] h_u h_v \, du \, dv = \int_{\partial R} a h_v \, dv - b h_u \, du. \tag{6.14}$$

The expression in brackets on the left is the divergence of the vector field. On the right the integrand measures the amount of the vector field crossing normal to the curve.

The next topic is the divergence theorem in three dimensions. Say that the vector field $\mathbf{v}$ has an expression in terms of normalized basis vectors of the form

$$\mathbf{v} = a\frac{1}{h_u}\frac{\partial}{\partial u} + b\frac{1}{h_v}\frac{\partial}{\partial v} + c\frac{1}{h_w}\frac{\partial}{\partial w}. \tag{6.15}$$

Recall that the volume form is

$$\mathrm{vol} = h_u h_v h_w \, du \, dv \, dw. \tag{6.16}$$

Then the corresponding differential 2-form is

$$\mathbf{v}\rfloor\mathrm{vol} = a h_v h_w \, dv \, dw + b h_w h_u \, dw \, du + c h_u h_v \, du \, dv. \tag{6.17}$$

The divergence theorem in three dimensions is obtained by applying Gauss's theorem for 2-forms to this particular 2-form. The result is

$$\int_V \left[\frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}(h_v h_w a) + \frac{\partial}{\partial v}(h_w h_u b) + \frac{\partial}{\partial w}(h_u h_v c)\right)\right] h_u h_v h_w \, du \, dv \, dw = \int_{\partial V} a h_v h_w \, dv \, dw + b h_w h_u \, dw \, du + c h_u h_v \, du \, dv. \tag{6.18}$$

The expression in brackets is the divergence of the vector field.

The divergence theorem works the same way in n dimensions. A vector field gives rise to an $n-1$ form whose differential is an n form. This n form is a scalar times the volume form.

The next topic is the classical Stokes's theorem in three dimensions. Say that the vector field $\mathbf{v}$ has an expression in terms of normalized basis vectors of the form

$$\mathbf{v} = a\frac{1}{h_u}\frac{\partial}{\partial u} + b\frac{1}{h_v}\frac{\partial}{\partial v} + c\frac{1}{h_w}\frac{\partial}{\partial w}. \tag{6.19}$$

Thus the vector field is expressed in terms of normalized basis vectors. Then there is a corresponding differential 1-form

$$\mathbf{gv} = ah_u\,du + bh_v\,dv + ch_w\,dw. \tag{6.20}$$

The classical Stokes's theorem in three dimensions is obtained by applying Stokes's theorem for 1-forms to this particular 1-form. This gives on the left hand side

$$\int_S \left[\frac{1}{h_v h_w}\left(\frac{\partial h_w c}{\partial v} - \frac{\partial h_v b}{\partial w}\right)\right] h_v h_w\,dv\,dw + \left[\frac{1}{h_u h_w}\left(\frac{\partial h_u a}{\partial w} - \frac{\partial h_w c}{\partial u}\right)\right] h_w h_u\,dw\,du \quad + \left[\frac{1}{h_u h_v}\left(\frac{\partial h_v b}{\partial u} - \frac{\partial h_u a}{\partial v}\right)\right] h_u h_v\,du\,dv \tag{6.21}$$

and on the right hand side

$$\int_{\partial S} ah_u\,du + bh_v\,dv + ch_w\,dw. \tag{6.22}$$

The terms in square brackets are the components of the curl of the vector field expressed in terms of normalized basis vectors. Notice that the fact that the curl may be expressed as a vector field depends on three dimensions. In general a vector field gives rise to a 1-form whose differential is a 2-form. In two dimensions this 2-form may be identified as a scalar times the area, while in three dimensions this 2-form may be associated with a vector field.

The situation with the cross product is similar. Two vectors give two 1-forms, whose exterior product is a 2-form. In two dimensions this gives a scalar, while in three dimensions it gives a vector.

Bibliography

[1] Ilka Agricola and Thomas Friedrich, *Global analysis: Differential forms in analysis, geometry and physics.*

[2] Henri Cartan, *Differential forms*, Houghton Mifflin, Boston, Massachusetts, 1970.

[3] Manfredo P. do Carmo, *Differential forms and applications*, Springer-Verlag, Berlin, 1994.

[4] Dominic G. G. Edelen, *Applied exterior calculus*, John Wiley & Sons, New York, 1985.

[5] Harold M. Edwards, *Differential forms: A differential forms approach*, Birkhäuser, Boston, 1969.

[6] Luther Pfahler Eisenhart, *Riemannian geometry*, Princeton University Press, Princeton, New Jersey, 1949.

[7] Harley Flanders, *Differential forms with applications to the physical sciences*, Academic Press, New York, 1963.

[8] John Hamal Hubbard and Barbara Burke Hubbard, *Vector calculus, linear algebra, and differential forms: A unified approach*, second ed., Prentice-Hall, Upper Saddle River, New Jersey, 2002.

[9] E. L. Ince, *Ordinary differential equations*, Dover Publications, New York, 1956.

[10] David Lovelock and Hanno Rund, *Tensors, differential forms, and variational principles*, Dover Publications, New York, 1989.

[11] John Milnor, *Morse theory*, Annals of Mathematics Studies, no. 51, Princeton University Press, Princeton, New Jersey, 1969.

[12] Edward Nelson, *Tensor analysis*, Princeton University Press, Princeton, NJ, 1967.

[13] ______, *Topics in dynamics I: Flows*, Princeton University Press, Princeton, NJ, 1969.

[14] M. Schreiber, *Differential forms: A heuristic introduction.*

[15] Michael Spivak, *Calculus on manifolds: A modern approach to classical theorems of advanced calculus*, Addison-Wesley, Reading, Massachusetts, 1965.

[16] Gabriel Weinreich, *Geometrical vectors*, University of Chicago Press, Chicago, 1998.

[17] Steven H. Weintraub, *Differential forms: A complement to vector calculus*, Academic Press, San Diego, 1997.