Obliquely Reflected Brownian motions (ORBMs) in Non-Smooth Domains

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Why Study Obliquely Reflected Diffusions?

Applications

- limits of interacting particle systems such as TASEP;
- diffusion approximations of stochastic networks;
- rank-dependent diffusion models (in finance);
- biological models of gene networks;
- closely related to certain (non-reflecting) diffusions;
Why Study Obliquely Reflected Diffusions?

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**Fundamental Mathematical Object**

- a (non-symmetric) Markov process in a domain
- many basic questions are still not fully understood
Focus of this talk ...

Obliquely reflected Brownian motions (ORBM)s in non-smooth (rough) domains, planar domains

Motivation

- Diffusions in fractal domains have unusual and interesting properties (Goldstein ('87), Kusuoka ('87), Barlow-Perkins ('88)
- loosely motivated by applications in biology
Existing techniques for

- reflected diffusions in piecewise smooth domains
- normally reflected diffusions in fractal domains

are not applicable
Draws on various joint works with

W. Kang
University of Maryland, Baltimore County
and
K. Burdzy, Z.-Q.-Chen and D. Marshall
University of Washington, Seattle
A Heuristic Description
Given a domain $D$ with a vector field $v(\cdot)$ on the boundary $\partial D$, ORBM behaves infinitesimally like Brownian motion in the interior, is constrained to stay within the closure $\bar{D}$ of the domain and spends zero Lebesgue time on the boundary.
Tools to study ORBMs in smooth domains

A. (Extended) Skorokhod problem (and stochastic differential equations with reflection – SDER)
B. Submartingale problem
A. The (Extended) Skorokhod Problem

The 1-dimensional Skorokhod Map \( D = [0, \infty), \nu(0) = e_1 \)

**Definition (Skorokhod Problem (Skorokhod '61))**

For every continuous \( \mathbb{R} \)-valued path \( \psi \), find a continuous path \( \phi \) s.t. \( \forall t \geq 0, \)

1. \( \phi(t) \geq 0 \) i.e., \( \phi(t) \) lies in \([0, \infty)\)
2. \( \eta = \phi - \psi \) is non-decreasing
3. “\( \eta \) increases only when \( \phi \) is on the boundary”

\[
\int_0^\infty \phi(s) d\eta(s) = 0.
\]
\[ \phi = \psi + \eta \geq 0, \quad \eta \text{ non-decreasing}, \quad \int_{0}^{\infty} \phi(s) \, d\eta(s) = 0. \]
An explicit formula (Skorokhod ’61)

\[ \phi = \psi + \eta \geq 0, \quad \eta \text{ non-decreasing}, \quad \int_0^\infty \phi(s) d\eta(s) = 0. \]

\[ \phi(t) = \Gamma_0(\psi)(t) = \psi(t) + \sup_{s \in [0,t]} [-\psi(s)]^+ \]

\[ Z = \Gamma_0(Z_0 + B) \text{ is RBM in 1-d} \]
Obtain reflected Brownian motion as a constrained version of Brownian motion.
Recall 1-d Skorokhod Problem (Skorokhod ’61)

For every continuous $\mathbb{R}$-valued path $\psi$, find a continuous $\phi$ s.t. $\forall t \geq 0$,

1. $\phi(t) \geq 0$ i.e., $\phi(t)$ lies in $[0, \infty)$;
2. $\eta = \phi - \psi$ is non-decreasing
3. “$\eta$ increases only when $\phi$ is on the boundary”

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\[
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1. Property 2 is equivalent to \( \eta(t) - \eta(s) \geq 0 \) for all \( 0 \leq s \leq t \);
Recall 1-d Skorokhod Problem (Skorokhod '61)

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3. “$\eta$ increases only when $\phi$ is on the boundary”

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\]

1. Property 2 is equivalent to $\eta(t) - \eta(s) \geq 0$ for all $0 \leq s \leq t$;
2. Setting $\nu(x) = 0$ if $x > 0$, properties 2 and 3 are equivalent to

\[
\eta(t) - \eta(s) \in \overline{co} \left( \cup_{u \in (s,t]} \nu(\phi(u)) \right),
\]

where, for $A \subset \mathbb{R}^d$, $\overline{co}[A]$ is the closure of the convex cone generated by the vectors in $A$. 
Obtain reflected Brownian motion as a constrained version of Brownian motion
Natural to consider piecewise smooth domains where $v$ is multi-valued

$$v(0) = \{\alpha_1 v_1 + \alpha_2 v_2 : \alpha_1, \alpha_2 \geq 0\}$$
The Extended Skorokhod Map on \((D, \nu(\cdot))\)

Extend \(\nu\) to \(\bar{D}\) by setting \(\nu(x) = 0\) for \(x \in D\)

**Definition (Extended Skorokhod Problem (’R ’06))**

For every continuous \(\mathbb{R}^d\)-valued path \(\psi\), find a continuous \(\phi\) s.t. \(\forall t \geq 0,\)

1. \(\phi(t) \in \bar{D}\);
2. \(\eta = \phi - \psi\) satisfies for every \(0 \leq s \leq t\),

\[
\eta(t) - \eta(s) \in \overline{co} \left( \bigcup_{u \in (s, t]} \nu(\phi(u)) \right),
\]

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**Previous Formulations and Results**

Tanaka (’79), Harrison-Reiman (’81), Lions-Sznitman (’84), Bernard El-Kharroubi (’91), Dupuis-Ishii (’91), Costantini (’92), Dupuis-Ramanan (’99), ...
Previous formulations of the Skorokhod Map assumed $\eta$ is of bounded variation;
This means that the RBM $Z = \Gamma(Z_0 + B)$ is always a semimartingale.
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The ESP can be used to construct both strong and weak solutions to the associated SDER.

B. The submartingale problem

The Submartingale Problem (Stroock-Varadhan ’71)

Given \((D, \nu(\cdot)), b, \sigma\), find probability measures \(Q_z, z \in \bar{D}\), on \(C([0, \infty) : \mathbb{R}^n)\) such that

- For every \(z \in \bar{D}\), \(Q_z(w(0) = z) = 1\);
- Under each \(Q_z\),

\[
M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s) \, ds
\]

is a submartingale for all \(f \in \mathcal{H}_0\), where

\[
\mathcal{H}_0 = \{f \in C_b^2(D) : \langle Df(x), \nu(x) \rangle \geq 0\}
\]
B. The submartingale problem

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\]

Well-posedness of the submartingale problem

The submartingale problem is said to be well posed if there exists a solution to the submartingale problem and it is unique.
1. Stroock and Varadhan (1971) established well-posedness of the submartingale problem for bounded $C^2$ domains with Lipschitz continuous reflection field $v$ satisfying $|\nabla v| \geq 1$.

2. Extended to specific non-smooth domains in Varadhan-Williams (1985); Williams (1987); Deblassie (1987, 1996); Deblassie-Toby (1993), ...
1. Stroock and Varadhan (1971) established well-posedness of the submartingale problem for bounded $C^2$ domains with Lipschitz continuous reflection field $v$ satisfying $|\nabla v| \geq 1$.

2. Extended to specific non-smooth domains in Varadhan-Williams (1985); Williams (1987); Deblassie (1987, 1996); Deblassie-Toby (1993), ...

3. A general multi-dimensional formulation was not available ... cited as an open problem (Williams 1995, DeBlassie 1997)
• The direction vector field $v(\cdot)$ can be multi-valued.
• For piecewise smooth domains $D = \cap_i D_i$, with each domain having a smooth vector field $v_i$, $v$ on the intersections of multiple smooth boundaries is defined as

$$v(x) = \left\{ \sum_{i \in I(x)} \alpha_i v_i(x), \alpha_i \geq 0 \right\},$$
A set $\mathcal{V}$ of irregular points where $\nu$ contains a half-plane

$$\mathcal{V} = \partial D \setminus \{x \in \partial D : \exists n \in n(x) \text{ such that } \langle n, \nu \rangle > 0, \forall \nu \in \nu(x)\}$$
Submartingale Formulation (Kang-'R, ’12)

Given \((D, v(\cdot)), b, \sigma\), find probability measures \(Q_z, z \in \bar{D}\), on \(\mathcal{C}([0, \infty) : \mathbb{R}^n)\) such that

- \(Q_z(\omega(0) = 0) = 1\)

\[ M_t^f = f(X_t) - \int_0^t \mathcal{L}f(X_s) \, ds, \quad t \geq 0, \]

is a \(Q_z\)-submartingale for all \(f \in \mathcal{H}\):

\[ \mathcal{H} \doteq \left\{ f \in C^2_c(\bar{D}) \oplus \mathbb{R} : \begin{array}{l} f \text{ is constant in a neighborhood of } V, \\ \langle v, \nabla f(y) \rangle \geq 0 \text{ for } v \in v(y) \text{ and } y \in \partial D \end{array} \right\} \]

For every \(z \in \bar{D}\), \(Q_z\)-almost surely,

\[ \mathcal{L}e b\{s \in [0, \infty) : \omega(s) \in V\} = 0. \]
Stroock and Varadhan (1969) introduced the martingale problem for diffusions and showed, under general conditions on $b$ and $\sigma$ that it was equivalent to the SDE formulation.

Reflected diffusions can also be defined as solutions to SDEs using the extended Skorokhod map $\Gamma$.

Is there a similar equivalence between SDEs and the submartingale formulation here?
Stroock and Varadhan (1969) introduced the martingale problem for diffusions and showed, under general conditions on $b$ and $\sigma$ that it was equivalent to the SDE formulation.

Reflected diffusions can also be defined as solutions to SDERs using the extended Skorokhod map $\Gamma$.

Is there a similar equivalence between SDERs and the submartingale formulation here?

Theorem (Kang-'R '13)

When the set $\mathcal{V}$ is finite, $D$ is piecewise $C^2$ and $\nu$ is $C^1$, $b$ bounded and measurable and $\sigma$ continuous, then well-posedness of submartingale formulation is equivalent to existence and uniqueness in law of weak solutions to SDERs.
Main thrust: need to construct a weak solution from a solution to the submartingale problem

For the martingale formulation, use test functions $f(x) = x_i$, $f(x) = x_i x_j$ to show that

$$f(X_t) - f(X_0) - \int_0^t \langle \nabla f(X_s), b(X_s) \rangle \, ds$$

is a martingale, then use the martingale representation theorem.
Main thrust: need to construct a weak solution from a solution to the submartingale problem

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Here, choice of test functions limited by derivative conditions

- Construction depends on geometry; especially at intersections of faces;
- need to identify the “local time” part;
- study behavior of quadratic variation of the mgale part of the Doob decomposition on the boundary;
Comments on the Proof

- Proof much more subtle ...

\[ D = \{ y \in \mathbb{R}^3 : y_2 \geq 0, L(y) \leq y_2 \leq R(y) \} \]

Suggests that additional conditions may need to be imposed near \( V \) ...

Are there a natural set of conditions ?

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Oblique Reflecting Brownian Motions
Proof much more subtle ...

In fact, the equivalence fails if $\mathcal{V}$ is not a finite set.

\[ D = \{ y \in \mathbb{R}^3 : y_2 \geq 0, L(y) \leq y_2 \leq R(y) \} \]

Suggests that additional conditions may need to be imposed near $\mathcal{V}$ ...

Are there a natural set of conditions?
Other Questions Related to Reflecting Diffusions

- semimartingale property
- characterization of stationary Distributions
- hitting edges and corners
- ...

K. Ramanan  Oblique Reflecting Brownian Motions
How can one even define normally reflected BMs in fractal domains?

**Challenge**
No way to make sense of the normal vector field
Normally Reflected ORBMs

Dirichlet form approach
Normally Reflected ORBMs

Dirichlet form approach

- \( E \) Hausdorff topological space, a Borel \( \sigma \)-field \( B(E) \), a \( \sigma \)-finite Borel measure \( m \);
Normally Reflected ORBMs

Dirichlet form approach

- $E$ Hausdorff topological space, a Borel $\sigma$-field $\mathcal{B}(E)$, a $\sigma$-finite Borel measure $m$;
- A pair $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, where
  - $\mathcal{D}(\mathcal{E})$ is a dense linear subspace of $L^2(\mathcal{E}; m)$;
  - $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \mapsto \mathbb{R}$ is a bilinear form;
Normally Reflected ORBM

Dirichlet form approach

- $E$ Hausdorff topological space, a Borel $\sigma$-field $\mathcal{B}(E)$, a $\sigma$-finite Borel measure $m$;
- A pair $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, where
  - $\mathcal{D}(\mathcal{E})$ is a dense linear subspace of $L^2(\mathcal{E}; m)$;
  - $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \mapsto \mathbb{R}$ is a bilinear form;
- that is symmetric: $\mathcal{E}(u, v) = \mathcal{E}(v, u), \forall u, v \in \mathcal{D}(\mathcal{E})$;
- and closed, i.e.,
  - $\mathcal{E}(u, u) \geq 0$;
  - $\mathcal{D}(\mathcal{E})$ is a Hilbert space when equipped with the norm $\mathcal{E}(u, v) + (u, v)_{L^2}$
- satisfies the contraction property

$$\mathcal{E}(u_*, u_*) \leq \mathcal{E}(u, u), \quad u_* = \min(\max(u, 0), 1).$$
The “energy” functional $\mathcal{E}$ is used to define a Markov process $X_t$, $t \geq 0$:

$$\mathcal{E}(u, u) = \lim_{t \downarrow 0} \frac{1}{2t} \int_E \mathbb{E}_z \left[ (u(X_t) - u(X_0))^2 \right] m(dz).$$

In the other direction,

$$(\mathcal{E}, \mathcal{D}(\mathcal{E}) \mapsto (T_t) \mapsto (\rho_t) \mapsto \{X_t\}_{t \geq 0}$$
The “energy” functional $\mathcal{E}$ is used to define a Markov process $X_t$, $t \geq 0$:

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$$

In the other direction,

$$(\mathcal{E}, \mathcal{D}(\mathcal{E}) \mapsto (T_t) \mapsto (p_t) \mapsto \{X_t\}_{t \geq 0}$$

Results using the Dirichlet Form Approach

- Beurling and Deny (1959)
- Silverstein and Fukushima (1970s)
- Fukushima (1990s): If a Dirichlet form on a locally compact state space is regular, one can construct an associated Markov process with RCLL paths.
Normal RBMs are symmetric Markov processes;
Dirichlet form techniques have been successively used to construct normally reflected Brownian motions in quite general domains (Z.-Q.-Chen '93)
Can the Dirichlet form approach be applied to ORBMs?

- ORBMs are not symmetric Markov processes;
Can the Dirichlet form approach be applied to ORBMs?

- ORBMs are not symmetric Markov processes;
- There exist extensions to the Dirichlet form approach that relax the symmetry assumption (sector condition);
Can the Dirichlet form approach be applied to ORBMs?

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- There exist extensions to the Dirichlet form approach that relax the symmetry assumption (sector condition);
- Ma and Röckner ('92): a more general result relating (non-symmetric) Dirichlet forms with Markov processes;
Can the Dirichlet form approach be applied to ORBMs?

- ORBMs are not symmetric Markov processes;
- There exist extensions to the Dirichlet form approach that relax the symmetry assumption (sector condition);
- Ma and Röckner ('92): a more general result relating (non-symmetric) Dirichlet forms with Markov processes;
- However, not much success with ORBMs.
Challenges

- The normal and tangential vector fields are not well defined in the classical sense
- ORBM is not a symmetric Markov process
- A new approach is required ...
ORBM in Smooth Planar domains

- Parametrize ORBMs in smooth domains by “angle of reflection”
- Let $B$ be standard two-dimensional Brownian motion
- Given $D \subset \mathbb{C}$ a smooth bounded open set and $\theta : \partial D \mapsto (-\pi/2, \pi/2)$ Borel measurable function satisfying $\sup_{x \in \partial D} |\theta(x)| < \pi/2$.
- $n(x)$: unit inward normal vector at $x \in \partial D$
- $t(x)$: unit tangent vector to $\partial D$ at $x$
- Vector field $v_\theta$ on $\partial D$ associated with $\theta$:
  \[ v_\theta(x) = n(x) + \tan \theta(x)t(x) \]
- Parametrize vector field in terms of the angle of reflection $\theta$
A. Domain Approximation

Given a simply connected Jordan domain $D$, $y_0 \in D$, approximate it by a sequence of smooth domains $D^k$ in the sense that for all $k$,

$$y_0 \in D_k \subset D_{k+1} \subset D,$$

and

$$\bigcup_k D_k = D$$

For each $k$ consider a smooth vector field $\theta^k$ and let $Z^k$ be ORBM associated with $(D^k, \theta^k)$. 
When does $Z^k$ converge to some limit process $Z$, and in what sense?
Is the “limit” $Z$ an ORBM in $D$ in any reasonable sense?
Is there an independent characterization of the limit ORBM?
Let $D_*$ denote the unit disc in the plane
Let $D_*$ denote the unit disc in the plane
First define the ORBM on the unit planar disc $D_*$
Then use conformal maps to extend the definition to more general domains
ORBM in the unit planar disc $D_*$

- Recall, given $\theta : \partial D \mapsto (-\pi/2, \pi/2)$,
  \[ v_\theta(x) = n(x) + \tan \theta(x)t(x) \]

- When $\theta$ is $C^2$, $D$ smooth, Skorokhod Map $\Gamma$ is well defined; RBM
  \[ Z = \Gamma(Z_0 + B) \]

  or, equivalently,
  \[ Z_t = Z_0 + B_t + \int_0^t v_\theta(Z_s) dL_s, \]

  Here, $L$ is the local time of $X$ on $\partial D$. 
ORBM in the unit planar disc $D_*$

- Recall, given $\theta : \partial D \mapsto (-\pi/2, \pi/2)$,
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  v_\theta(x) = n(x) + \tan \theta(x)t(x)
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  \[
  Z = \Gamma(Z_0 + B)
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  or, equivalently,
  \[
  Z_t = Z_0 + B_t + \int_0^t v_\theta(Z_s)dL_s,
  \]
  Here, $L$ is the local time of $X$ on $\partial D$.

- When $D = D_*$, strong solution exists without regularity assumption on $\theta$ (use polar decomposition)
B. Conformal Mapping

\[ \mathcal{T} = \{ \theta \in L^\infty(\partial D_*) : \|\theta\|_\infty \leq \pi/2, \ \theta \neq \pi/2, \ \text{and} \ \theta \neq -\pi/2 \}. \]

Suppose \( \theta \in \mathcal{T} \) is \( C^2 \) and let \( Z \) be a \((D_*, v_\theta)\) ORBM.
B. Conformal Mapping

\[ \mathcal{T} = \{ \theta \in L^\infty(\partial D_\ast) : \|\theta\|_\infty \leq \pi/2, \; \theta \neq \pi/2, \; \text{and} \; \theta \neq -\pi/2 \}. \]

Suppose \( \theta \in \mathcal{T} \) is \( C^2 \) and let \( Z \) be a \((D_\ast, v_\theta)\) ORBM.

- Let \( D \) be a simply connected bounded domain
- Let \( f : D_\ast \mapsto D \) be a one-to-one onto analytical function.
B. Conformal Mapping

\[ \mathcal{T} = \{ \theta \in L^\infty(\partial D_*) : ||\theta||_{\infty} \leq \pi/2, \ \theta \neq \pi/2, \ \text{and} \ \theta \neq -\pi/2 \} \]

Suppose \( \theta \in \mathcal{T} \) is \( C^2 \) and let \( Z \) be a \((D_*, v_\theta)\) ORBM.

- Let \( D \) be a simply connected bounded domain
- Let \( f : D_* \mapsto D \) be a one-to-one onto analytical function.
- Define

\[
c(t) = \int_0^t |f'(Z_s)|^2 ds, \quad \text{for } t \geq 0,
\]
\[
\zeta = \inf\{ t \geq 0 : c(t) = \infty \},
\]
\[
Y(t) = f(Z_{c^{-1}(t)}), \quad \text{for } t \in [0, \zeta).
\]
**B. Conformal Mapping (contd.)**

- \( Z \) be a \((D_*, \nu_\theta)\) ORBM.
- Let \( f : D_* \mapsto D \) be a one-to-one onto analytical function.
- \( Y(t) = f(Z_{c^{-1}}(t)), \quad \text{for } t \in [0, \zeta) \).
- Then \( Y \) is an extension of killed Brownian motion in \( D \), i.e., for every \( t \geq 0 \) and \( \tau_t = \inf\{s \geq t : Y_s \in \partial D\} \), the process \( \{Y_s, s \in [t, \tau_t)\} \) is Brownian motion killed upon exiting \( D \).
B. Conformal Mapping (contd.)

- $Z$ be a $(D_*, v_\theta)$ ORBM.
- Let $f : D_* \mapsto D$ be a one-to-one onto analytical function.
- $Y(t) = f(Z_{c^{-1}(t)})$, for $t \in [0, \zeta)$.
- Then $Y$ is an extension of killed Brownian motion in $D$, i.e., for every $t \geq 0$ and $\tau_t = \inf\{s \geq t : Y_s \in \partial D\}$, the process $\{Y_s, s \in [t, \tau_t]\}$ is Brownian motion killed upon exiting $D$.
- Is $Y$ an ORBM in a suitable sense?
Results: A. Smooth domain approximation

\( D \subset \mathbb{R}^2 \) – open bounded simply connected set

\( D_k \subset D_{k+1}, \bigcup_k D_k = D \), \( D_k \) have smooth boundaries

\( \theta_k(x) \) – reflection angle; \( x \in \partial D_k \)

\( Z^k \) – obliquely reflected Brownian motion in \( D_k \)
Results: A. Smooth domain approximation

\[ D \subset \mathbb{R}^2 \] – open bounded simply connected set
\[ D_k \subset D_{k+1}, \bigcup_k D_k = D, \ D_k \text{ have smooth boundaries} \]
\[ \theta_k(x) \] – reflection angle; \( x \in \partial D_k \)
\[ Z^k \] – obliquely reflected Brownian motion in \( D_k \)

**THEOREM** (forthcoming; Burdzy, Chen, Marshall, ’R)

Suppose that, as \( k \to \infty \), \( \theta_k : \partial D_* \mapsto (-\pi/2, -\pi/2) \) converges to \( \theta \) in the weak-* topology (as elements of the dual space of \( \mathbb{L}^1(\partial D_*) \)). Then obliquely reflected Brownian motions \( Z^k \) converge, as \( k \to \infty \), to a process \( Z \) in \( D \).
Results: A. Smooth domain approximation

\( D \subset \mathbb{R}^2 \) – open bounded simply connected set

\( D_k \subset D_{k+1}, \bigcup_k D_k = D, D_k \) have smooth boundaries

\( \theta_k(x) \) – reflection angle; \( x \in \partial D_k \)

\( Z^k \) – obliquely reflected Brownian motion in \( D_k \)

THEOREM (forthcoming; Burdzy, Chen, Marshall, ’R)

Suppose that, as \( k \rightarrow \infty \), \( \theta_k : \partial D_\ast \mapsto (-\pi/2, -\pi/2) \) converges to \( \theta \) in the weak-* topology (as elements of the dual space of \( L^1(\partial D_\ast) \)). Then obliquely reflected Brownian motions \( Z^k \) converge, as \( k \rightarrow \infty \), to a process \( Z \) in \( D \).

How does one independently characterize the ORBM?
Jumps on the boundary when $\theta(x) = \pi/2$

- Limit process could have jumps

- So convergence of $Z^k$ to $Z$ is (in general) in a certain $M_1$ topology
Jumps on the boundary when $\theta(x) = \pi/2$

- Limit process could have jumps

- So convergence of $Z^k$ to $Z$ is (in general) in a certain $M_1$ topology
- Limit could be "excursion reflected Brownian motion (ERBM)" when limit $\theta \equiv \pi/2$
Towards an independent characterization

Alternative parametrization of ORBMs in $D_*$

$D_*$ – unit disc in $\mathbb{R}^2$

$\theta(x)$ – angle of reflection at $x \in \partial D$

$$T = \{ \theta \in L^\infty(\partial D_*) : ||\theta||_\infty \leq \pi/2, \, \theta \not\equiv \pi/2, \, \text{and} \, \theta \not\equiv -\pi/2 \}.$$
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$$\theta \in \mathcal{T} \leftrightarrow (h, \mu_0) \in \mathcal{H} \times \mathbb{R}$$

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$h(x)dx$ – stationary distribution

$\mu_0$ – “rate of rotation” of $Z$ around zero
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$h(x)dx -$ stationary distribution

$\mu_0 -$ “rate of rotation” of $Z$ around zero

and
Rate of Rotation $\mu_0$

$D_*$ – unit disc in $\mathbb{R}^2$, $\theta(x)$ – angle of reflection at $x \in \partial D$

$$\theta \leftrightarrow (h, \mu_0)$$

$h(x)dx$ – stationary distribution

$\mu_0$ – rate of rotation around zero

$$\lim_{t \to \infty} \frac{1}{t} \arg X_t - \mu_0 \Rightarrow \text{Cauchy}.$$
Alternative Parametrization of ORBMs

\[ \theta \leftrightarrow (h, \mu_0) \]

The correspondence has quite an explicit form

**THEOREM** (forthcoming; Burdzy, Chen, Marshall, ’R)

\[
h(z) = \frac{\text{Re} \exp(\tilde{\theta}(z) - i\theta(z))}{\pi \text{Re} (e^{-i\theta(0)})} = \frac{\text{Re} \exp(\tilde{\theta}(z) - i\theta(z))}{\pi \cos \theta(0)}, \quad z \in D,
\]

\[
\mu_0 = \tan \theta(0) = \int_D \tan \theta(z) h(z) dz,
\]

\[
\theta(z) = -\arg \left( h(z) + i\tilde{h}(z) - i\mu/\pi \right), \quad z \in D.
\]
We can also parameterize the ORBM in terms of “rotation rates”

\[ \theta \in \mathcal{T} \leftrightarrow \mu(\cdot) \in \mathcal{R}, \]

\[ \mathcal{R} = \{ \mu : \mu \text{ is harmonic in } D_\ast \text{ and } \tilde{\mu}(z) > -1, \text{ for all } z \in D_\ast \}. \]

Again, \( \mu(z) \) can be written quite explicitly in terms of \( \theta \).

Probabilistic interpretation of \( \mu(z) \):

\[ \lim_{t \to \infty} \arg^* \frac{X_t - z}{t} = \mu(z). \]
Theorem (forthcoming; Burdzy, Chen, Marshall, ’R)

For any simply connected bounded domain $D$ and $\theta \in \mathcal{T}$, we can define an ORBM $Y$ using a conformal mapping $f$ as described earlier.

If $\theta \leftrightarrow (h, \mu)$, $\theta \leftrightarrow \mu$, then $Y$ has stationary density

$$\hat{h} = h \circ f^{-1} / ||h \circ f^{-1}||^D_1,$$

and

$$\lim_{t \to \infty} \arg^* \frac{Y_t - z}{t} = \frac{\mu(f^{-1}(z))}{||h \circ f^{-1}||^D_1}.$$  \hspace{1cm} (1)

For any $\mu_0 \in \mathbb{R}$ and $\hat{h}$ a positive harmonic function in $D$ with $||\hat{h}||_1 = 1$, there exists an ORBM $Y$ in $D$ such that $Y$ has stationary distribution $\hat{h}$ and (1) holds with $\mu(\cdot) \leftrightarrow (\mu_0, h)$. 
THEOREM (forthcoming; Burdzy, Chen, Marshall, ’R)

Suppose that, as \( k \to \infty \), \( \theta_k : \partial D_* \mapsto (\frac{-\pi}{2}, \frac{-\pi}{2}) \) converges to \( \theta \) in the weak-* topology (as elements of the dual space of \( \mathbb{L}^1(\partial D_*) \)). Then obliquely reflected Brownian motions \( Z^k \) converge, as \( k \to \infty \), to a process \( Z \) in \( D \).

[Recall: forthcoming; Burdzy, Chen, Marshall, ’R] The limit process \( Z \) can be characterized in terms of \( \theta \) and \( D \) as described above. And this characterization is consistent with the ORBM obtained in terms of the conformal mapping.
Reflected Diffusions in piecewise-smooth domains arise in a variety of fields ranging from math physics and finance to queueing theory.

A new paradigm has been developed for characterization of ORBMs in bounded planar domains, including some ORBMs with jumps (excursion-reflected Brownian motions).

Many questions remain regarding the construction of ORBMs in more general (multiply connected) planar domains as well as multi-dimensional domains.

Several foundational questions remain even for RBMs in polyhedral domains.
List of Some of My Relevant Works