The Growth Model: Busemann Functions, Shape, Geodesics, and Other Stories

Firas Rassoul-Agha

Department of Mathematics
University of Utah

May 20, 2014

Joint work with Nicos Georgiou (Sussex) and Timo Seppäläinen (Madison-Wisconsin)
Summary/Conclusion

We consider the growth model (oriented last passage percolation) on the two-dimensional square lattice.

Want to describe the large-scale behavior of the model.

In the special case of exponential or geometric weights, explicit computations are possible and many results exist.

Our long-term goal is to develop an approach for general weight distribution.

**Busemann functions** play a central role: boundary weights for stationary processes (like invariant measures are for Markov chains), infinite geodesics, shape formula, solve variational formulas, tool for proving results about fluctuation exponents.
Summary/Conclusion

In geometry: Busemann functions are used to study large-scale geometry in metric spaces.

In first passage percolation (random geometry on $\mathbb{Z}^2$): introduced by Newman in mid ’90ies then Hoffman in ’08 to study infinite geodesics.

Licea-Newman ’96 proved almost-sure existence of Busemanns by first proving coalescence of directional geodesics, under a global (unproven) curvature assumption.

Recently Damron-Hanson used a weak limit to get a version of Busemann functions and show existence of directional infinite geodesics in first passage percolation under weaker assumptions than those of Newman: differentiability or strict convexity of the shape is enough.

For the growth model, we prove almost sure existence of Busemann functions, then describe the shape function and geodesics under similar assumptions to those of Damron-Hanson.
Last Passage Percolation

Random potential $\omega = (\omega_x)_{x \in \mathbb{Z}^2} \in \Omega$, $\mathbb{R}$-valued i.i.d., $2 + \varepsilon$ moments.

Think of $\omega_x$ as time spent at $x$ (even though they can be negative).

Up-right paths $x_{0,n} = (x_0, \ldots, x_n)$ take steps $e_1 = (1, 0)$ or $e_2 = (0, 1)$.

Passage time of path $x_{0,n}$ is $\sum_{i=0}^{n-1} \omega_{x_i}$. 
Last Passage Percolation

Random potential $\omega = (\omega_x)_{x \in \mathbb{Z}^2} \in \Omega$, $\mathbb{R}$-valued i.i.d., $2 + \varepsilon$ moments.

Think of $\omega_x$ as time spent at $x$ (even though they can be negative).

Up-right paths $x_{0,n} = (x_0, \ldots, x_n)$ take steps $e_1 = (1, 0)$ or $e_2 = (0, 1)$.

Passage time of path $x_{0,n}$ is $\sum_{i=0}^{n-1} \omega_{x_i}$.

Point-to-point last passage time: $G_{x,y}(\omega) = \max_{x_{0,n}} \sum_{x_0=x, x_n=y}^{n-1} \omega_{x_k}$.

Point-to-level last passage time: $G_{(n)}(\omega) = \max_{x_{0,n}: x_0=0} \sum_{k=0}^{n-1} \omega_{x_k}$.
Shape Theorem

LLN says sum of i.i.d. grows linearly.

$G_{0,x}$ is not quite a sum of i.i.d.

It is however superadditive: $G_{x,y} + G_{y,z} \leq G_{x,z}$.

Then: outside one null set, for each $\xi \in \mathbb{R}^2_+$

$$g_{pp}(\xi) = \lim_{n \to \infty} n^{-1} G_{0,[n\xi]}$$

exists almost-surely, is concave, homogenous ($g_{pp}(c\xi) = cg_{pp}(\xi)$) and continuous all the way to the boundary.
Shape Theorem

LLN says sum of i.i.d. grows linearly.

$G_{0,x}$ is not quite a sum of i.i.d.

It is however superadditive: $G_{x,y} + G_{y,z} \leq G_{x,z}$.

Then: outside one null set, for each $\xi \in \mathbb{R}^2_+$

$$g_{pp}(\xi) = \lim_{n \to \infty} n^{-1} G_{0,[n\xi]}$$ exists almost-surely, is concave, homogenous ($g_{pp}(c\xi) = cg_{pp}(\xi)$) and continuous all the way to the boundary.
Path $x_{0,n}$ that maximizes $G_{x,y}$ is called a geodesic.

Unique if $\omega_0$ is continuous. Otherwise, can talk e.g. about topmost and rightmost geodesics.

This way we have a tree of geodesics from 0.

By compactness, at least one infinite geodesic exists.
Path \( x_0,n \) that maximizes \( G_{x,y} \) is called a geodesic.

Unique if \( \omega_0 \) is continuous. Otherwise, can talk e.g. about topmost and rightmost geodesics.

This way we have a tree of geodesics from 0.

By compactness, at least one infinite geodesic exists.

But is there an infinite geodesic in any given direction \( \xi \in \mathcal{U} = \{(t, 1 - t) : t \in (0, 1)\} \)?

Is it the limit of the geodesic from 0 to \( [n\xi] \) as \( n \to \infty \)?

If \( \omega_0 \) is continuous, are these \( \xi \)-geodesics unique?

Do \( \xi \)-geodesics out of \( x \) and \( y \) coalesce (i.e. eventually merge)?
Licea and Newman ’96: answers are in the positive for standard first passage percolation (nearest-neighbor paths minimizing the passage time) if $g_{pp}(\xi)$ satisfies a global curvature assumption.

Problem: the curvature assumption has not been proved. Though conjectured to hold.

Damron and Hanson ’14: Existence holds under just strict convexity or differentiability of $g_{pp}$ (which should be “easier” to prove).

Ferrari and Pimentel ’05: answers are in the positive also for the last passage percolation model we are considering, but with $\omega_0$ exponential.

The exponential model is one of the solvable models for which explicit computations are possible. In particular, an explicit formula is available for the shape $g_{pp}(\xi)$.

Would like to allow more general weight distributions.
Fluctuations

CLT says that if $X_{0,n}$ has increments $e_1$ or $e_2$ equally likely, then it fluctuates from its average (straight line from 0 to $(n/2, n/2)$) by $n^{1/2}$.

Limit distribution of $(X_n - (n/2, n/2))/n^{1/2}$ is Gaussian.
Fluctuations

CLT says that if $X_{0,n}$ has increments $e_1$ or $e_2$ equally likely, then it fluctuates from its average (straight line from 0 to $(n/2, n/2)$) by $n^{1/2}$.

Limit distribution of $(X_n - (n/2, n/2))/n^{1/2}$ is Gaussian.

Say $\omega_0$ is continuous.

What are the fluctuations of the geodesic from 0 to $[n\xi]$?

What about geodesic from 0 to level $n$? (i.e. the path maximizing $G(n)$)

Conjecture: with enough moments on $\omega_0$ (Auffinger and Louidor '11) geodesic fluctuations are of order $n^{2/3}$.

Superdiffusivity is because the path goes “out of its way” looking for high values of the potential.
Fluctuations

CLT says that if $X_{0,n}$ has increments $e_1$ or $e_2$ equally likely, then it fluctuates from its average (straight line from 0 to $(n/2, n/2)$) by $n^{1/2}$.

Limit distribution of $(X_n - (n/2, n/2))/n^{1/2}$ is Gaussian.

Say $\omega_0$ is continuous.

What are the fluctuations of the geodesic from 0 to $[n\xi]$?

What about geodesic from 0 to level $n$? (i.e. the path maximizing $G(n)$)

Conjecture: with enough moments on $\omega_0$ (Auffinger and Louidor '11) geodesic fluctuations are of order $n^{2/3}$.

Superdiffusivity is because the path goes “out of its way” looking for high values of the potential.

On the other hand, $G_{0,[n\xi]}$ (and $G(n)$) should have $n^{1/3}$ fluctuations.

Limit distributions related to Tracy-Widom from random matrices.
Models with these fluctuation exponents are said to belong to the Kardar-Parisi-Zhang (KPZ) universality class.

Johansson ’00 proved LPP with exponential weights is in the KPZ class.

Again: solvability of the model was key.
Understanding the shape

Consider a finite subset $V \subset \mathbb{Z}^2$ containing 0 (e.g. $\{u : |u| \leq L\}$).

$\{G_{0,[n\xi]} - u - G_{0,[n\xi]} : u \in V\}$ describes the microscopic shape around $[n\xi]$.

We expect this random vector to converge in distribution as $n \to \infty$.

Shifting by $-[n\xi]$ and reflecting $\omega_x \mapsto \omega_{-x}$ turns the above into $\{G_{u,[n\xi]} - G_{0,[n\xi]} : u \in V\}$.

Now maybe even almost sure convergence holds.
Understanding the shape

Consider a finite subset $V \subset \mathbb{Z}^2$ containing 0 (e.g. $\{u : |u| \leq L\}$).

$\{ G_0,[n\xi] - u - G_0,[n\xi] : u \in V \}$ describes the microscopic shape around $[n\xi]$.

We expect this random vector to converge in distribution as $n \to \infty$.

Shifting by $-[n\xi]$ and reflecting $\omega_x \mapsto \omega_{-x}$ turns the above into $\{ G_u,[n\xi] - G_0,[n\xi] : u \in V \}$.

Now maybe even almost sure convergence holds.

**Busemann functions:** $B^\xi(x, y; \omega) = \lim_{n \to \infty} (G_{x,[n\xi]} - G_{y,[n\xi]})$.

Limit exists if e.g. geodesics coalesce.
Understanding geodesics

Note that $G_{x,[n\xi]} = \omega_x + \max(G_{x+e_1,[n\xi]}, G_{x+e_2,[n\xi]})$.

So $(G_{x,[n\xi]} - G_{x+e_1,[n\xi]}) \land (G_{x,[n\xi]} - G_{x+e_2,[n\xi]}) = \omega_x$ almost surely.

At each point $x$, geodesic to $[n\xi]$ follows the smallest of the two gradients.
Understanding geodesics

Note that \( G_{x,[n\xi]} = \omega_x + \max(G_{x+e_1,[n\xi]}, G_{x+e_2,[n\xi]}) \).

So \( (G_{x,[n\xi]} - G_{x+e_1,[n\xi]}) \land (G_{x,[n\xi]} - G_{x+e_2,[n\xi]}) = \omega_x \) almost surely.

At each point \( x \), geodesic to \([n\xi]\) follows the smallest of the two gradients.

\( n \to \infty \) gives \( B^\xi(x, x + e_1) \land B^\xi(x, x + e_2) = \omega_x \) almost surely.

The above suggests that \( \xi \)-geodesic out of \( x \) should follow the smallest \( B^\xi(x, x + z) \), \( z \in \{e_1, e_2\} \).
Understanding geodesics

Note that $G_{x,[n\xi]} = \omega_x + \max(G_{x+e_1,[n\xi]}, G_{x+e_2,[n\xi]})$.

So $(G_{x,[n\xi]} - G_{x+e_1,[n\xi]}) \wedge (G_{x,[n\xi]} - G_{x+e_2,[n\xi]}) = \omega_x$ almost surely.

At each point $x$, geodesic to $[n\xi]$ follows the smallest of the two gradients.

$n \to \infty$ gives $B^{\xi}(x, x + e_1) \wedge B^{\xi}(x, x + e_2) = \omega_x$ almost surely.

The above suggests that $\xi$-geodesic out of $x$ should follow the smallest $B^{\xi}(x, x + z), \; z \in \{e_1, e_2\}$.

If a tie occurs, then can go either way:

Always breaking ties with $e_1$ should give the rightmost $\xi$-geodesic

Always breaking ties with $e_2$ should give the topmost $\xi$-geodesic
Busemann functions

Consider $g_{pp}$ as a concave function on $\mathcal{U} = \{(t, 1 - t) : t \in (0, 1)\}$.

Given $\xi \in \mathcal{U}$ at which $g_{pp}$ is differentiable, let $[\underline{\xi}, \bar{\xi}] \subset \mathcal{U}$ be the maximal (possibly degenerate) interval containing $\xi$ on which $g_{pp}$ is linear.

Standing assumptions:

$\mathbb{P}\{\omega_0 \geq c\} = 1$,
$\omega_x$ i.i.d. with $2 + \epsilon$ moments,
$\xi \in \mathcal{U}$, and $\xi$ and $\bar{\xi}$ are points of differentiability.

Theorem. $B_{\xi}(x, y; \omega) = \lim_{n \to \infty} (G_{x, [n \xi]} - G_{y, [n \xi]})$ exists almost surely.

Furthermore: Same limit for all directions in the same linear segment.

Corollary. If $g_{pp}$ is differentiable, limits exist $\forall \xi$. (No convexity needed.)

Remark. $\omega_0 \geq c$ only because we use results from queuing where service times were assumed nonnegative. All the queuing results seem to go through without this assumption.
Busemann functions

Consider $g_{pp}$ as a concave function on $\mathcal{U} = \{(t, 1 - t) : t \in (0, 1)\}$.

Given $\xi \in \mathcal{U}$ at which $g_{pp}$ is differentiable, let $[\underline{\xi}, \overline{\xi}] \subset \mathcal{U}$ be the maximal (possibly degenerate) interval containing $\xi$ on which $g_{pp}$ is linear.

**Standing assumptions:** $\mathbb{P}\{\omega_0 \geq c\} = 1$, $\omega_x$ i.i.d. with $2 + \varepsilon$ moments, $\xi \in \mathcal{U}$, and $\underline{\xi}$ and $\overline{\xi}$ are points of differentiability.

**Theorem.** $B^{\xi}(x, y; \omega) = \lim_{n \to \infty} (G_{x,[n\xi]} - G_{y,[n\xi]})$ exists almost surely. Furthermore: Same limit for all directions in the same linear segment.

**Corollary.** If $g_{pp}$ is differentiable, limits exist $\forall \xi$. (No convexity needed.)
Busemann functions

Consider \( g_{pp} \) as a concave function on \( \mathcal{U} = \{(t, 1-t) : t \in (0, 1)\} \).

Given \( \xi \in \mathcal{U} \) at which \( g_{pp} \) is differentiable, let \([\underline{\xi}, \overline{\xi}] \subset \mathcal{U}\) be the maximal (possibly degenerate) interval containing \( \xi \) on which \( g_{pp} \) is linear.

Standing assumptions: \( \mathbb{P}\{\omega_0 \geq c\} = 1, \omega_x \) i.i.d. with \( 2 + \varepsilon \) moments, \( \xi \in \mathcal{U} \), and \( \underline{\xi} \) and \( \overline{\xi} \) are points of differentiability.

**Theorem.** \( B^\xi(x, y; \omega) = \lim_{n \to \infty} (G_{x,[n\xi]} - G_{y,[n\xi]}) \) exists almost surely.

Furthermore: Same limit for all directions in the same linear segment.

**Corollary.** If \( g_{pp} \) is differentiable, limits exist \( \forall \xi \) . (No convexity needed.)

**Remark.** \( \omega_0 \geq c \) only because we use results from queuing where service times were assumed nonnegative. All the queuing results seem to go through without this assumption.
Properties

$L^1$: $\mathbb{E}[|B^\xi(x, y)|] < \infty$.

Stationary: $B^\xi(x, y; T_z\omega) = B^\xi(x + z, y + z; \omega)$ ($(T_z\omega)_x = \omega_{x+z}$)

Cocycle: $B^\xi(x, y) + B^\xi(y, z) = B^\xi(x, z)$.

The space of $L^1$ stationary cocycles: $\mathcal{C}$.

For $B \in \mathcal{C}$ we can find $h(B) \in \mathbb{R}^2$ such that $\mathbb{E}[B(x, y)] = h(B) \cdot (x - y)$. 
Properties

$L^1$: $\mathbb{E}[|B^\xi(x, y)|] < \infty$.

Stationary: $B^\xi(x, y; T_z \omega) = B^\xi(x + z, y + z; \omega)$ ($\left( T_z \omega \right)_x = \omega_{x+z}$)

Cocycle: $B^\xi(x, y) + B^\xi(y, z) = B^\xi(x, z)$.

The space of $L^1$ stationary cocycles: $\mathcal{C}$.

For $B \in \mathcal{C}$ we can find $h(B) \in \mathbb{R}^2$ such that $\mathbb{E}[B(x, y)] = h(B) \cdot (x - y)$.

Potential recovery: $B^\xi(0, e_1) \wedge B^\xi(0, e_2) = \omega_0$ almost surely.

Monotonicity: $\xi \cdot e_1 > \zeta \cdot e_1$ ($\xi$ to the right of $\zeta$) $\Rightarrow$

$$B^\xi(0, e_1) \leq B^\zeta(0, e_1) \text{ and } B^\xi(0, e_2) \geq B^\zeta(0, e_2)$$

($G_{0,[n\xi]} - G_{e_1,[n\xi]} \leq G_{0,[n\zeta]} - G_{e_1,[n\zeta]}$ is forced by planarity.)

(One consequence will be: $\xi$-geodesic is to the right of $\zeta$-geodesic.)
If $B \in \mathcal{C}$ then a $B$-geodesic is a path that follows the minimal $B(x, x + z)$, $z \in \{e_1, e_2\}$. (In case of ties, it can go either way.)

**Theorem.** If $B$ recovers potential $\omega (B(0, e_1) \land B(0, e_2) = \omega_0$ a.s.) then a $B$-geodesic is a geodesic: every finite piece of it is a geodesic.
Geodesics

If $B \in \mathcal{C}$ then a $B$-geodesic is a path that follows the minimal $B(x, x + z)$, $z \in \{e_1, e_2\}$. (In case of ties, it can go either way.)

**Theorem.** If $B$ recovers potential $\omega$ ($B(0, e_1) \wedge B(0, e_2) = \omega_0 \text{ a.s.}$) then a $B$-geodesic is a geodesic: every finite piece of it is a geodesic.

Given $\xi$ where $g_{pp}$ is differentiable, recall $[\underline{\xi}, \bar{\xi}]$. A geodesic $x_{0,\infty}$ is directed in $[\underline{\xi}, \bar{\xi}]$ if all limit points of $x_n/n$ belong to this interval.
Geodesics

If $B \in \mathcal{C}$ then a $B$-geodesic is a path that follows the minimal $B(x, x + z)$, $z \in \{e_1, e_2\}$. (In case of ties, it can go either way.)

**Theorem.** If $B$ recovers potential $\omega$ ($B(0, e_1) \wedge B(0, e_2) = \omega_0$ a.s.) then a $B$-geodesic is a geodesic: every finite piece of it is a geodesic.

Given $\xi$ where $g_{pp}$ is differentiable, recall $[\underline{\xi}, \bar{\xi}]$. A geodesic $x_{0,\infty}$ is directed in $[\underline{\xi}, \bar{\xi}]$ if all limit points of $x_n/n$ belong to this interval.

**Theorem.**

a) Any $B^\xi$-geodesic is directed in $[\underline{\xi}, \bar{\xi}]$.

b) Any geodesic directed in $[\underline{\xi}, \bar{\xi}]$ is a $B^\xi$-geodesic.

c) The $B^\xi$-geodesic with $e_2$-tie breaks is the topmost of all geodesics directed in $[\underline{\xi}, \bar{\xi}]$. Similarly for rightmost.
**Corollary.** If \( g_{pp} \) is differentiable everywhere, then every geodesic is directed in \([\xi, \bar{\xi}]\) for some \( \xi \).

**Remark.** Can also handle corners using geodesics ordering, but will omit.

Thus can show: If \( g_{pp} \) is strictly concave, then every geodesic has a direction \( \xi \), i.e. \( \lim x_n/n \) exists almost surely.
Corollary. If \( g_{pp} \) is differentiable everywhere, then every geodesic is directed in \([\xi, \xi]\) for some \( \xi \).

Remark. Can also handle corners using geodesics ordering, but will omit.

Thus can show: If \( g_{pp} \) is strictly concave, then every geodesic has a direction \( \xi \), i.e. \( \lim x_n/n \) exists almost surely.

Theorem. Assume also \( P\{\omega_0 \leq r\} \) is continuous in \( r \). Then \( P\{B^{\xi}(0, e_1) = B^{\xi}(0, e_2)\} = 0 \).

Corollary. If \( \omega_0 \) is continuous, then there exists a unique geodesic directed in \([\xi, \xi]\) out of every point \( x \in \mathbb{Z}^2 \).

Theorem. Topmost \([\xi, \xi]\)-directed geodesics coalesce, rightmost \([\xi, \xi]\)-geodesics coalesce, and when \( \omega_0 \) is continuous, \([\xi, \xi]\)-geodesics coalesce.
Variational formula

Until recently, the only description of $g_{pp}(\xi)$ was from superadditivity: 

$$g_{pp}(\xi) = \sup_n n^{-1} \mathbb{E}[G_0, [n\xi]].$$

Going through random polymer models:

**Theorem.** $g_{pp}(\xi) = \inf_{B \in \mathcal{C}} \text{ess sup}\{\omega_0 - B(0, e_1; \omega) \wedge B(0, e_2; \omega) - h(B) \cdot \xi\}.$

(Recall: $\mathcal{C}$ is class of $L^1$ stationary cocycles, $\mathbb{E}[B(x, y)] = h(B) \cdot (x - y).$)

This is a similar formula to the one in Arjun Krishnan’s earlier talk on first passage percolation.

Such formulas are important in statistical mechanics: their solutions are expected to describe the infinite-volume system (i.e. geodesics and shape as $n \to \infty$).
**Theorem.** Under the standing assumptions, $B^\xi$ solves the variational formula for $g_{pp}(\xi)$. In fact, the essential supremum is not needed and we have almost surely

$$g_{pp}(\xi) = \omega_0 - B^\xi(0, e_1, \omega) \wedge B^\xi(0, e_2, \omega) - h(B^\xi) \cdot \xi.$$ 

(In homogenization theory, a solution like $B^\xi$ that removes the need for $\text{ess sup}$ is called a corrector.)

**Corollary.** Due to potential recovery, we have $g_{pp}(\xi) = -h(B^\xi) \cdot \xi$.

Using some calculus one then gets that $h(B^\xi) = -\nabla g_{pp}(\xi)$.

Nice interpretation: average microscopic gradient is macroscopic gradient.
When $\omega_0$ are exponential or geometric we in fact can calculate explicitly the distributions of $B^\xi(0, e_1)$ and $B^\xi(0, e_2)$ for all $\xi = (\xi_1, \xi_2)$.

For example, if $\omega_0$ is exponential with rate 1, then $B^\xi(0, e_1)$ is exponential with rate $\alpha$ and $B^\xi(0, e_2)$ is exponential with rate $1 - \alpha$, where

$$\alpha = \frac{\sqrt{\xi_1}}{\sqrt{\xi_1} + \sqrt{\xi_2}}.$$

Then $h(B^\xi) = -\left(\mathbb{E}[B^\xi(0, e_1)], \mathbb{E}[B^\xi(0, e_2)]\right) = -\left(\frac{1}{\alpha}, \frac{1}{1-\alpha}\right)$

and $g_{pp}(\xi) = -h(B^\xi) \cdot \xi = \xi_1 + \xi_2 + 2\sqrt{\xi_1 \xi_2}$.

This is the known formula derived by Rost '81.
When $\omega_0$ is exponential or geometric, $B^\xi(ne_1, (n + 1)e_1)$ are i.i.d. and so are $B^\xi(ne_2, (n + 1)e_2)$.

Balázs, Cator, and Seppäläinen '06 used this to prove the $n^{2/3}$ fluctuations of the geodesic and $n^{1/3}$ fluctuations of the last passage time, in the exponential weights case, with less technology that Johansson's proof of the Tracy-Widom limit.

More generally, CLT exponents for fluctuations of $B^\xi(0, ne_1)$ and $B^\xi(0, ne_2)$ imply information about fluctuation exponents of last passage quantities. (The above BCS result is one way to achieve this.)

Now we have a promising route to proving universality of KPZ fluctuations (i.e. for general weight distributions).
Competition interface

For simplicity, assume $\omega_0$ is continuous.

Consider the geodesics tree out of 0 and its subtrees out of $e_1$ and $e_2$.

The two subtrees are separated by a competition interface.
Competition interface

For simplicity, assume $\omega_0$ is continuous.

Consider the geodesics tree out of 0 and its subtrees out of $e_1$ and $e_2$.

The two subtrees are separated by a competition interface.

**Theorem.** Assume $g_{pp}$ differentiable at endpoints of its linear segments.

a) The competition interface almost surely has a direction $\xi^*(\omega) \in U$.

b) $\xi^*$ is supported everywhere outside the segments on which $g_{pp}$ is linear.

c) In this (random) direction $\xi^*(\omega)$, there are two geodesics out of 0, one through $e_1$ and one through $e_2$.

(This generalizes part of Coupier’s ’11 results for exponential $\omega_0$ to general weights. When $\omega_0$ is exponential, there are no linear segments.)
Direction $\xi_*(\omega)$ has a nice characterization.

By monotonicity, $B^\xi(0, e_1) - B^\xi(0, e_2)$ decreases as $\xi \cdot e_1$ increases.

We have seen: $\forall \xi \in \mathcal{U}$ rational, $\mathbb{P}\{B^\xi(0, e_1) = B^\xi(0, e_2)\} = 0$.

Can show difference goes to $\infty$ as $\xi \to e_2$ and to $-\infty$ as $\xi \to e_1$.

Hence, $\exists! \xi_*(\omega)$ s.t.

$B^\xi(0, e_1) > B^\xi(0, e_2)$ ($\xi$-geodesics go to $e_2$) if $\xi \cdot e_1 < \xi_* \cdot e_1$ and

$B^\xi(0, e_1) < B^\xi(0, e_2)$ ($\xi$-geodesics go to $e_1$) if $\xi \cdot e_1 > \xi_* \cdot e_1$.

$\xi_*$ is the direction of the competition interface.
When $\omega_0$ is exponential (e.g. with rate 1) $B^\xi(0, e_1)$ and $B^\xi(0, e_2)$ are independent exponentials with known rates.

Then we can compute the distribution of $\xi_\ast$ explicitly.

Switching to the angle $\theta_\ast$ that $\xi_\ast$ makes with $e_1$ we recover Ferrari and Pimentel’s result:

$$P\{\theta_\ast \leq t\} = \frac{\sqrt{\sin t}}{\sqrt{\sin t + \sqrt{\cos t}}} \quad t \in [0, \pi/2].$$

Can also deal with the case when $\omega_0$ is not continuous, and then get explicit results for $\omega_0$ geometric.
Existence of Busemann functions

First, we get the existence of a process \( \{ B^\xi(x, y; \omega, \hat{\omega}) : \xi \in U, x, y \in \mathbb{Z}^2 \} \) on a larger (product) probability space.

This process is such that \( B^\xi \) is an \( L^1 \) stationary cocycle that recovers the original potential \( \omega \), and satisfies the monotonicity Busemann functions are supposed to satisfy.

This step is done by appealing to a fixed point result for a certain queuing operator from queuing theory.

Then, we couple gradients \( G_{x,[n\xi]} - G_{y,[n\xi]} \) with that process, sandwiching them between \( B^\zeta \) and \( B^\eta \) (plus a small error), where \( \zeta \cdot e_1 < \xi \cdot e_1 < \eta \cdot e_1 \).

This allows us to prove that the limiting Busemann function exists (at \( \xi \) as in standing assumption) and equals the \( B^\xi \) from queuing. In particular, there was no need to extend the space.
Thank You