# REMARKS ON THE MEMBRANE AND BUCKLING EIGENVALUES FOR PLANAR DOMAINS

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Abstract. I present a counter-example to the conjecture that the first eigenvalue of the clamped buckling problem in a planar domain is not smaller than the third eigenvalue of the fixed membrane in that domain. I also prove that the conjecture holds for domains that are invariant under rotation by angle $\pi/2$.

## 1. Introduction

Let $\Omega$ be a bounded planar domain with smooth boundary $\Gamma$. By $\lambda_1(\Omega) < \lambda_2(\Omega) \leq \cdots$ I denote the eigenvalues of the Dirichlet Laplacian in $\Omega$ (the membrane eigenvalues;) each eigenvalue is repeated as many times as its multiplicity is. Let $\beta_1(\Omega) \leq \beta_2(\Omega) \leq \cdots$ be the values of the parameter $\beta$, for which the buckling problem

$$\begin{cases} \Delta^2 w(x) + \beta \Delta w(x) = 0 & \text{in } \Omega \\ w(x) = \frac{\partial w(x)}{\partial \nu} = 0 & \text{on } \Gamma \end{cases} \tag{1.1}$$

has a non-trivial solution. Here $\nu(x)$, $x \in \Gamma$, is the outward unit normal vector to $\Gamma$. With some abuse of terminology, I call $\beta_j(\Omega)$ an eigenvalue of the buckling problem in $\Omega$, and a non-trivial solution of (1.1) will be called an eigenfunction.

Payne proved in [P] that

$$\beta_1(\Omega) \geq \lambda_2(\Omega), \tag{1.2}$$

and he suggested that the even sharper inequality $\beta_1(\Omega) \geq \lambda_3(\Omega)$ may hold. In this paper, I present a counter-example to the last conjecture.

**Theorem 1.** *There exists a convex planar domain $\Omega$ with smooth boundary such that $\beta_1(\Omega) < \lambda_3(\Omega)$.*

Theorem 1 is proved in section 2 by looking at the buckling and membrane eigenvalues of a deformed circle. For a circle, the first buckling eigenvalue coincides with the second membrane eigenvalue, wich is of multiplicity two. It turns out that, for almost every deformation of a circle, the multiple eigenvalue of the Dirichlet Laplacian splits in such a way that the first buckling eigenvalue becomes strictly in between the second and the third membrane eigenvalues. The typical graphs of

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the buckling and membrane eigenvalues as functions of the deformation parameter can be seen in Fig. 1.

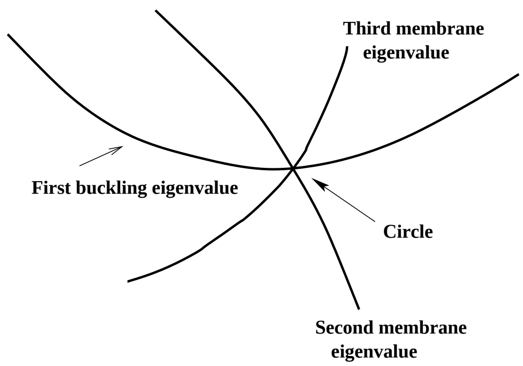


**Fig. 1:** The buckling and membrane eigenvalues for a deformed circle

The second theorem that I prove says that the conjecture holds for domains with a rotational $\mathbb{Z}/4\mathbb{Z}$ symmetry.

**Theorem 2.** *Let a bounded planar domain $\Omega$ be invariant under rotation by the angle $\pi/2$. Then $\beta_1(\Omega) \geq \lambda_3(\Omega)$.*

Let me first remind the reader that the inequalities $\beta_j(\Omega) > \lambda_j(\Omega)$ follow immediately from variational characterization of the Dirichlet and buckling eigenvalues. The buckling eigenvalues can be determined by applying min-max formulae (e.g., see [CH]) to the quotient

$$R_b(w) = \frac{\int_\Omega |\Delta w(x)|^2 dx}{\int_\Omega |\nabla w(x)|^2 dx},$$

with $w(x)$ taken from the Sobolev space $H_0^2(\Omega)$. The eigenvalue problem for the Dirichlet Laplacian is equivalent to $\Delta^2 u + \lambda \Delta u = 0$, with the boundary conditions $u(x) = \Delta u(x) = 0$ on $\Gamma$. From this fact, one can easily conclude that the same min-max formulae give the Dirichlet eigenvalues; however the test functions shoud be taken now from the space $H^2(\Omega) \cap H_0^1(\Omega)$. Because the latter space is bigger than $H_0^2(\Omega)$, one gets smaller values for $\lambda_j(\Omega)$.

**Remark.** Unfortunately, Payne's proof of inequality (1.2) contains a gap. Let $w_1(x)$ be a non-trivial solution of (1.1) with $\beta = \beta_1(\Omega)$, and let $u_1(x)$ be the eigenfunction of the Dirichlet Laplacian that corresponds to the smallest eigenvalue $\lambda_1(\Omega)$. To prove (1.2), Payne uses test functions $\psi_j(x) = a_j w_1(x) + \partial w_1(x)/\partial x_j$, $j = 1, 2$, that are orthogonal to $u_1(x)$ (see (34) and (35) in [P].) Such functions clearly exist if

$$\int_\Omega w_1(x) u_1(x) \neq 0. \tag{1.3}$$

However, it is not clear why (1.3) should hold. It is known that the ground state of the buckling problem is not necessarily positive (see [AL] and references there.) Numerical calculations in [W] show that it is not positive even for a square. If there is no positivity for $w_1(x)$ then (1.3) becomes questionable.

Fortunately, Payne's proof can be easily repaired.[1] In the case when

$$\int_\Omega w_1(x)u_1(x) = 0,$$

one takes a two-dimensional space $L$ of functions spanned by $u_1(x)$ and $w_1(x)$. All functions from $L$ belong to $H^2(\Omega)\cap H_0^1(\Omega)$. Let us estimate the Rayleigh quotient $R_b(\psi)$ for a function $\psi = aw_1 + bu_1$ from $L$. One has

$$\int_\Omega \Delta w_1(x)u_1(x)dx = -\int_\Omega \nabla w_1(x)\cdot\nabla u_1(x) = -\lambda_1\int_\Omega w_1(x)u_1(x)dx = 0,$$

and therefore

$$\begin{aligned}\int_\Omega |\Delta\psi|^2dx &= \int_\Omega |a\Delta w_1 - \lambda_1 bu_1|^2dx = |a|^2\int_\Omega |\Delta w_1|^2dx + \lambda_1^2|b|^2\int_\Omega |u_1|^2dx\\ &= \mu_1|a|^2\int_\Omega |\nabla w_1|^2dx + \lambda_1|b|^2\int_\Omega|\nabla u_1|^2dx\\ &\le \mu_1\int_\Omega [|a|^2|\nabla w_1|^2 + |b|^2|\nabla u_1|^2]dx = \mu_1\int_\Omega|\nabla\psi|^2dx.\end{aligned}$$

The Rayleigh quotient $R_b(\psi)$ does not exceed $\mu_1$ for all functions $\psi$ from a two-dimentional subspace $L$ of $H^2(\Omega)\cap H_0^1(\Omega)$; so $\lambda_2(\Omega)\le\mu_1(\Omega)$ in the case when (1.3) does not hold as well.

## 2. Proof of Theorem 1

A domain $\Omega$, for which $\beta_1 < \lambda_3$, will be a small perturbation of the unit disk $D = \{x : |x| < 1\}$. The eigenfunctions and the eigenvalues of the Dirichlet Laplacian in the disk are well known. One has $\lambda_2(D) = \lambda_3(D) = j_{1,1}^2$ where $j_{1,1}$ is the smallest positive zero of the Bessel function $J_1(r)$. The corresponding eigenspace $E$ is two-dimensional; functions $v_1(x) = cJ_1(j_{1,1}r)\sin\theta$ and $v_2(x) = cJ_1(j_{1,1}r)\cos\theta$ form an ortho-normal basis in $E$. Here $(r,\theta)$ are polar coordinates, and the constant $c$ is chosen from the condition

$$1 = \int_D v_j(x)^2dx = \pi c^2\int_0^1 rJ_1(j_{1,1}r)^2dr = \frac{\pi c^2}{j_{1,1}^2}\int_0^{j_{1,1}} rJ_1(r)^2dr.$$

Therefore,

$$c^2 = \frac{j_{1,1}^2}{\pi\int_0^{j_{1,1}} rJ_1(r)^2dr}. \tag{2.1}$$

[1]R. Laugesen told me that he knew about the gap in Payne's proof in 1994; he found then a slightly different way of filling the gap.

It can be easily computed that the first eigenvalue of the buckling problem for the disk, $\beta_1(D)$, equals the same $j_{1,1}^2$, it is simple, and the corresponding eigenfunction

$$w(x) = J_0(j_{1,1}r) - J_0(j_{1,1}). \tag{2.2}$$

The boundary conditions are satisfied because $J_0' = -J_1$. I do not normalize $w(x)$.

Let $\alpha(\theta)$ be a smooth function on the unit circle $S$, and let $\Omega(\tau) = \{x : 0 \le r < 1+\tau\alpha(\theta)\}$. If $|\tau|$ is small enough, then $\Omega(\tau)$ is a convex planar domain with smooth boundary. For small values of $\tau$, the double eigenvalue $\lambda_2 = \lambda_3$ of the Dirichlet Laplacian in $D$ splits into two $\tau$-analytic functions $\lambda_2 + \mu_j\tau + \cdots$, $j = 1, 2$. The numbers $\mu_j$ are eigenvalues of the matrix $A = (a_{ij})$, with

$$a_{ij} = -\int_S \frac{\partial v_i}{\partial r}\frac{\partial v_j}{\partial r}\alpha(\theta)d\theta$$

(see [GSch], [M].) If

$$\int_0^{2\pi} \alpha(\theta)\sin(2\theta)d\theta \ne 0, \tag{2.3}$$

then the matrix $A$ is not scalar, $\mu_1 \ne \mu_2$, and, for small positive values of $|\tau|$, $\lambda_2(\Omega(\tau)) \ne \lambda_3(\Omega(\tau))$. Let

$$f(\tau) = \frac{\lambda_2(\Omega(\tau)) + \lambda_3(\Omega(\tau))}{2}.$$

Then

$$f'(0) = \frac{1}{2}\mathrm{tr}A = -\frac{1}{2}c^2 j_{1,1}^2 J_1'(j_{1,1})^2 \int_0^{2\pi} \alpha(\theta)d\theta. \tag{2.4}$$

Let $\beta(\tau) = \beta_1(\Omega(\tau))$. For small values of $|\tau|$, the function $\beta(\tau)$ is real-analytic. Indeed, one can construct a family of diffeomorphisms $\Phi(\tau) : \Omega(\tau) \to D$ that is real-analytic in $\tau$. They induce a family of Riemannian metrics $\Phi(\tau)_*(\delta_{ij})$ on the disk $D$. These metrics give rise to Laplace-Beltrami operators $\Delta(\tau)$ in $D$, and, finally, $\beta_j(\tau)^{-1}$ are eigenvalues of the operator

$$T(\tau) = ((\Delta(\tau)^2)_D)^{-1/2}\Delta(\tau)((\Delta(\tau)^2)_D)^{-1/2}.$$

Here $(\Delta(\tau)^2)_D$ is the bi-Laplacian with the Dirichlet boundary conditions $u = \partial u/\partial\nu = 0$ on the boundary $S$. The family of self-adjoint operators $T(\tau)$ is real-analytic in $\tau$; the standard perturbation theory (e.g., see [K]) tells us that, for small values of $|\tau|$, the ground state is simple, and $\beta(\tau)$ is real-analytic in $\tau$. Moreover, one can construct a family of eigenfunctions $w(\tau)$ of the buckling problem that correspond to the smallest eigenvalue $\beta(\tau)$ and that is real-analytic in $\tau$. Functions $w(\tau)$ can be chosen to be real-valued.

My goal now is to compute $\beta'(0)$. To do the computation, one can transplant the problem to the disk, and then use the Rayleigh formula from the perturbation theory. This approach is rather straightforward, but it is somewhat time consuming. I use a different approach, similar to the one used in [GSch]. I assume that $\alpha(\theta) \ge 0$.

Then, for positive values of $\tau$, all eigenfunctions $w(\tau)$ are defined in the disk $D$. Differentiating the equation (1.1) with respect to $\tau$ and setting $\tau = 0$, one gets

$$\Delta^2 w'(0) + \beta_1(D)\Delta w'(0) + \beta'(0)\Delta w = 0. \tag{2.5}$$

Here, by $'$ I denote the $\tau$-derivative. The equation (2.5) is valid in $D$. I multiply (2.5) by $w$, integrate over $D$, use Green's formula, and use the equation (1.1) to get

$$\int_S \frac{\partial w'(0)}{\partial r}\Delta w d\theta - \int_S w'(0)\frac{\partial \Delta w}{\partial r} d\theta - \beta'(0)\int_D |\nabla w|^2 dx = 0. \tag{2.6}$$

Notice that $w'(0)$ vanishes on $S$. Indeed, both $w(\tau)$ and its gradient vanish at $r = 1 + \tau\alpha(\theta)$. Therefore, $w(\tau)(1, \theta) = O(\tau^2)$, and $w'(0)(1, \theta) = 0$. So, the second term in (2.6) vanishes. Then, for positive small values of $\tau$,

$$\frac{\partial w(\tau)}{\partial r}(1, \theta) = -\tau\alpha(\theta)\frac{\partial^2 w(\tau)}{\partial r^2}(1 + \tau\alpha(\theta), \theta) + O(\tau^2).$$

One divides the last equality by $\tau$ and takes the limit $\tau \to 0$ to get

$$\frac{\partial w'(0)}{\partial r}(1, \theta) = -w_{rr}(1, \theta)\alpha(\theta). \tag{2.7}$$

Obviously,

$$\Delta w(1, \theta) = w_{rr}(1, \theta). \tag{2.8}$$

Equations (2.6)–(2.8) lead us to the formula

$$\beta'(0) = -\frac{\int_S |w_{rr}(1, \theta)|^2 \alpha(\theta) d\theta}{\int_D |\nabla w|^2 dx} \tag{2.9}$$

By substituting the explicit formula (2.2) for $w(x)$ into (2.9) and by using $J_0'(r) = -J_1(r)$, one gets

$$\beta'(0) = -\frac{j_{1,1}^4 |J_1'(j_{1,1})|^2}{2\pi \int_0^{j_{1,1}} r J_1^2(r) dr} \int_0^{2\pi} \alpha(\theta) d\theta.$$

Now, compare the last formula with (2.4) and (2.1) to see that

$$\beta'(0) = f'(0) = \frac{\mu_1 + \mu_2}{2}.$$

If the condition (2.3) is satisfied then $\mu_1 \neq \mu_2$, and one of the $\mu$'s, say $\mu_1$, is bigger than $\beta'(0)$. For small positive values of $\tau$, the third eigenvalue of the Dirichlet Laplacian is $j_{1,1}^2 + \mu_1\tau + O(\tau^2)$, and it is bigger than the first eigenvalue of the buckling problem, which equals $j_{1,1}^2 + \beta'(0)\tau + O(\tau^2)$. Note that the same thing happens for small negative values of $\tau$; then $j_{1,1}^2 + \mu_2\tau + O(\tau^2)$ is the third eigenvalue of the Dirichlet Laplacian. We see that, as long as (2.3) is satisfied, any small deformation of the disk gives a counter-example to the conjectue $\beta_1 \geq \lambda_3$.

**Remark.** R. Laugesen noticed that, for constructing a counter-example, one can avoid computing $\beta'(0)$ for a general function $\alpha(\theta)$. Take $\alpha(\theta) = \sin(2\theta)$. Then $f'(0) = 0$ (see (2.4).) The counter-clockwise rotation by angle $\pi/2$ about the origin maps $\Omega(\tau)$ onto $\Omega(-\tau)$ because $\alpha(\theta + \pi/2) = -\alpha(\theta)$. Therefore, $\beta(-\tau) = \beta(\tau)$, and $\beta'(0) = 0 = f'(0)$. The condition (2.3) clearly holds for $\sin(2\theta)$. This argument works for functions of the type

$$\alpha(\theta) = \sum_{k=0}^{\infty}(a_k \cos((4k + 2)\theta) + b_k \sin((4k + 2)\theta))$$

with $b_0 \neq 0$.

## 3. Proof of Theorem 2

Let $p$ be the counter-clockwise rotation of the plane $\mathbb{R}^2$ about the origin by angle $\pi/2$. The space $L^2(\Omega)$ splits into the direct sum of spaces $L^2_{\pm 1}(\Omega)$ and $L^2_{\pm i}(\Omega)$ where $L^2_\zeta(\Omega) = \{u(x) \in L^2(\Omega) : u(p(x)) = \zeta u(x)\}$. These spaces are invariant under both the Dirichlet Laplacian and the buckling operator, so the spectra of both problems split into the union of their spectra in four symmetry sectors. I denote the corresponding eigenvalues by $\lambda_{j,\zeta}$ and $\beta_{j,\zeta}$ where $\zeta = \pm 1, \pm i$. Notice that $\overline{L^2_i(\Omega)} = L^2_{-i}(\Omega)$, both the Dirichlet Laplacian and the buckling problem are invariant under complex conjugation; therefore $\lambda_{j,i} = \lambda_{j,-i}$ and $\beta_{j,i} = \beta_{j,-i}$. Notice also that the proof of $\beta_j > \lambda_j$ works for each symmetry sector separately; so $\beta_{j,\zeta}(\Omega) > \lambda_{j,\zeta}(\Omega)$.

*Case 1.* $\beta_1(\Omega) = \beta_{1,\pm i}(\Omega)$.

The first eigenvalue of the Dirichlet Laplacian belongs to the $L^2_1$ sector because of positivity of the ground state. Therefore, in this case,

$$\lambda_1(\Omega) = \lambda_{1,1}(\Omega) < \lambda_{1,\pm i}(\Omega) < \beta_{1,\pm i}(\Omega) = \beta_1(\Omega),$$

and there are at least three eigenvalues of the Dirichlet Laplacian, $\lambda_1$ and $\lambda_{1,i} = \lambda_{1,-i}$, that are smaller than $\beta_1$.

*Case 2.* $\beta_1(\Omega) = \beta_{1,1}(\Omega)$ or $\beta_1(\Omega) = \beta_{1,-1}(\Omega)$.

In both cases, the ground state of the buckling problem, $w_1(x)$, can be taken as a real-valued, even function with respect to central symmetry $o = p^2$. All its directional derivatives $\partial w_1/\partial\omega$ are odd functions with respect to $o$; therefore they are orthogonal to $u_1(x)$. For a unit vector $\omega$, I denote $\tilde\omega = p(\omega)$. One has

$$\beta_1(\Omega) = \frac{\int_\Omega (\Delta w_1)^2 dx}{\int_\Omega |\nabla w_1|^2 dx} = \frac{\int_\Omega \left( \left|\nabla \frac{\partial w_1}{\partial \omega}\right|^2 + \left|\nabla \frac{\partial w_1}{\partial \tilde\omega}\right|^2 \right) dx}{\int_\Omega \left( \left(\frac{\partial w_1}{\partial \omega}\right)^2 + \left(\frac{\partial w_1}{\partial \tilde\omega}\right)^2 \right) dx}.$$

Notice that $\partial w_1/\partial\tilde\omega = \pm\partial w_1/\partial\omega$; therefore,

$$\beta_1(\Omega) = \frac{\int_\Omega \left|\nabla \frac{\partial w_1}{\partial \omega}\right|^2 dx}{\int_\Omega \left(\frac{\partial w_1}{\partial \omega}\right)^2 dx},$$

and the Rayleigh quotient

$$R(v) = \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega |v|^2 dx}$$

equals $\beta_1(\Omega)$ for any function from a two-dimensional space $\{v(x) = a\partial w_1/\partial x_1 + b\partial w_1/\partial x_2\} \subset H^1_0(\Omega)$ that is orthogonal to $u_1(x)$. Therefore, $\beta_1(\Omega) \geq \lambda_3(\Omega)$.

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