EXTREMAL PROPERTIES OF EIGENVALUES FOR A METRIC GRAPH

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1. Introduction

Let Γ be a connected finite graph; by V we denote the set of its vertices, and by E we denote the set of its edges. If each edge e is considered as a segment of certain length $l(e) > 0$ then such a graph is called a *metric graph.* One can find a good survey and numerous references in [K]. A metric graph with a given combinatorial structure Γ is determined by a vector of edge lengths $(l(e)) \in \mathbb{R}_+^{|E|}$. We will use the notation $G = (\Gamma, (l(e)))$. *The length of a metric graph*, $l(G)$, is the sum of the lengths of all its edges. Sometimes, it is convenient to treat each edge as a pair of oriented edges; then, on an oriented edge, one defines a coordinate x_e that runs from 0 to $l(e)$. If $-e$ is the same edge, with the opposite orientation, then $x_{-e} = l(e) - x_e$. If an edge e emanates from a vertex v, we express it by writing $v \prec e$.

A function ϕ on G is a collection of functions $\phi_e(x)$ defined on each edge e. We say that it belongs to $L^2(G)$ if each function ϕ_e belongs to L^2 on the corresponding edge; then

$$||\phi||^2 = \sum_e ||\phi_e||^2.$$

The Sobolev space $H^1(G)$ is defined as the space of continuous functions on G that belong to H^1 on each edge. The Laplacian on G is defined via the quadratic form

$$\int_G |\phi'(x)|^2 dx = \sum_{e\in E} \int_0^{l(e)} |\phi_e'(x)|^2 dx$$

considered on the natural domain $H^1(G)$. The Laplacian Δ is given by the differential expression $-d^2/dx_e^2$ on each edge. Its domain is the set of continuous functions that belong to the Sobolev space H^2 on each edge and that satisfy the Kirchhoff condition

$$\sum_{e\succ v} \frac{d\phi}{dx_e}(v) = 0 \tag{1.1}$$

The first version of this paper dealt with the smallest positive eigenvalue only; the proof was completely different. Y. Colin de Verdiere and S. Gallot suggested the use of the symmetrization technique. As a result, the theorem became more general and the proof became simpler. My great thanks to them.

for every vertex v. This operator is self-adjoint, and its spectrum consists of eigenvalues

$$0 = \mu_1(G) < \mu_2(G) \le \mu_3(G) \le \cdots \nearrow \infty$$

of finite multiplicity. The eigenvalues are the numbers for which the problem

$$\frac{d^2\phi_e}{dx_e^2} + \lambda\phi_e = 0, \tag{1.2}$$

subject to the Kirchhoff conditions (1.1), has a non-trivial solution. For the sake of brevity, we will call $\{\mu_j(G)\}$ the spectrum of the metric graph G.

In this paper, we study the extremal properties for $\mu_j(G)$ in the class of metric graphs with a fixed length l. First, let us make explicit computations for three simple examples.

Example 1. Γ is a cyclic graph with k vertices $v_1, \dots, v_k$. It has k edges that connect v_1 with v_2, v_2 with v_3, $\dots$, v_k with v_1. Obviously, the spectrum of the Laplacian on such a graph is the same as the spectrum of the Laplacian on a circle of circumference $l = l(G)$, so

$$\mu_1(G) = 0, \quad \mu_{2k}(G) = \mu_{2k+1}(G) = 4\pi^2k^2l(G)^{-2}, \quad k \ge 1. \tag{1.3}$$

Example 2. Γ is a linear graph with k vertices. It is the same graph as in the previous example, with the edge connecting v_k and v_1 removed. The spectrum of the Laplacian on such a graph coincides with the spectrum of the Neumann Laplacian on the interval $[0, l]$, so

$$\mu_k(G) = \pi^2(k-1)^2l(G)^{-2}. \tag{1.4}$$

Example 3. Γ is a star with k edges. It has $k+1$ vertices $v_0, v_1, \dots, v_k$, and v_0 is connected with all other vertices. We assume that $k \ge 2$; in the case when $k = 2$, Γ is a linear graph. For a metric graph $G = H_k$, we take the lengths of all edges to be equal to l/k. Let us orient an edge e_j that connects v_j with v_0 toward v_0. Then an eigenfunction of the Laplacian on e_j must be of the form $a_j\cos(\sqrt{\lambda}x_j)$ because it satisfies the Neumann condition at $x_j = 0$. If $l\sqrt{\lambda}/k \ne -(\pi/2) + \pi m$, $m \in \mathbb{Z}_+$, then this function does not vanish at v_0, all a_j must be equal to each other, and the Kirchhoff condition (1.1) is satisfied if $\sin(l\sqrt{\lambda}/k) = 0$, or $l\sqrt{\lambda}/k = \pi m$, $m \in \mathbb{Z}_+$. One gets a family of simple eigenvalues $\pi^2k^2m^2/l(G)^2$, $m \in \mathbb{Z}_+$, of the Laplacian. If $l\sqrt{\lambda}/k = -(\pi/2) + \pi m$ then the function vanishes at v_0, and it is continuous for all values of a_j. The Kirchhoff condition at v_0 is equivalent to $a_1 + \cdots + a_k = 0$. Therefore,

$$\lambda = \pi^2k^2(2m-1)^2/4l(G)^2, \quad m \in \mathbb{Z}_+,$$

are also eigenvalues of the Laplacian; their multiplicity equal $k-1$. We see that, for a star,

$$\mu_2(H_k) = \mu_k(H_k) = \frac{\pi^2k^2}{4l(H_k)^2}. \tag{1.5}$$

The third example shows that, in the class of metric graphs of fixed length, $\mu_2(G)$, and, therefore, $\mu_j(G)$, $j \ge 2$, does not admit an upper bound. The best lower bound for $\mu_j(G)$, $j \geq 2$, can be seen when $G = H_j$. The main purpose of this paper is to prove that, in fact, the smallest possible value for $\mu_j(G)$ is achieved when $G = H_j$.

Obviously, one can always remove vertices of degree 2 from the list of vertices. To make some statements simpler, from this point, we assume that there are no vertices of degree 2 in G.

Theorem 1. *Let G be a connected metric graph. Then*

$$\mu_j(G) \geq \frac{\pi^2 j^2}{4l(G)^2}, \quad j \geq 2. \tag{1.6}$$

Moreover, an equality in (1.6) occurs if and only if G is a segment when $j = 2$ and $G = H_j$ when $j \geq 3$.

Remark. It is known that, in the class of bounded, connected planar domains of given area, Ω, the first eigenvalue $\lambda_1(\Omega)$ of the Dirichlet Laplacian in Ω is minimized when Ω is a circle, and the first positive eigenvalue $\mu_2(\Omega)$ of the Neumann Laplacian in Ω assumes its maximal value when Ω is a circle [PS]. Moreover $\lambda_1(\Omega)$ can be arbitrarily big, and $\mu_2(\Omega)$ can be arbitrarily close to 0. Though it may look like the eigenvalues of a metric graph should be analogues of the eigenvalues of the Neumann Laplacian: the domain of the Dirichlet functional in the variational formulation is the whole space $H^1(G)$, their extremal properties are closer to those of the eigenvalues of the Dirichlet Laplacian.

2. Proof of Theorem 1

First, it is sufficient to prove the inequality in Theorem 1 for trees. In fact, let G be a metric graph, and let G' be the graph that is obtained from G by cutting an edge e at some point x_0. This point gives rise to two different vertices in G'. Obviously, $H^1(G) \subset H^1(G')$, so $\mu_j(G) \geq \mu_j(G')$ because $\mu_j(G)$ is obtained by the min-max principle from the Rayleigh quotient over a smaller space. If G is not a tree, one can cut several edges of G to make a connected tree out of it, and the j-th eigenvalue of that tree does not exceed $\mu_j(G)$.

Let G be a connected metric tree. By $\phi_1(x) = \text{const}$, $\phi_2(x), \dots$, we denote the eigenfunctions of the Laplacian on G that correspond to the eigenvalues $\mu_1 = 0$, $\mu_2, \dots$. Fix an integer $j \geq 2$. For any collection of points $x_1, \dots, x_m \in G$, $m \leq j-1$, one can find a non-zero linear combination, $\phi(x)$, of $\phi_1(x), \dots, \phi_j(x)$ that vanishes at all those points. One has

$$\int_G |\phi'(x)|^2 dx \leq \mu_j(G) \int_G |\phi(x)|^2 dx. \tag{2.1}$$

The set $G \setminus \{x_1, \dots, x_m\}$ consists of a certain number of connected components. By $G(x_1, \dots, x_m)$ we denote the disjoint union of their closures. Each connected component of $G(x_1, \dots, x_m)$ is a tree. Let us formulate the first lemma that we need.

Lemma 2. *Let G be a connected metric tree, and let $j \geq 2$ be an integer. Then there exist points $x_1, \dots, x_m$, $m \leq j - 1$, such that the length of each connected component of $G(x_1, \dots, x_m)$ does not exceed $l(G)/j$.*

2A. Proof of (1.6) from Lemma 2. We choose points $x_1, \dots, x_m$ from Lemma 2. Then, for at least one of the connected components of $G(x_1, \dots, x_m)$ (we call it G_1,) $\phi(x)$ is not identically 0 on G_1, and

$$\int_{G_1} |\phi'(x)|^2 dx \le \mu_j(G) \int_{G_1} |\phi(x)|^2 dx. \tag{2.2}$$

When restricted to G_1, the function $\phi(x)$ satisfies the Dirichlet boundary condition at one of its leaves. The next lemma gives a lower bound for the ground state of the Laplacian with the Dirichlet condition at a point.

For a metric graph G and a point $y \in G$, we denote by $H^1_y(G)$ the space of $H^1(G)$ functions that vanish at y.

Lemma 3. *Let G be a connected metric graph and $y \in G$. Then*

$$\int_G |\phi'(x)|^2 dx \ge \frac{\pi^2}{4l(G)^2} \int_G |\phi(x)|^2 dx \tag{2.3}$$

for all functions $\phi \in H^1_y(G)$. For a non-zero function $\phi \in H^1_y(G)$, the equality in (2.3) may happen only if G is a segment, y is its endpoint, and $\phi(x)$ is proportional to $\sin(\pi s/(2l(G))$ where s is the distance to y.

One obtains the inequality in Theorem 1 by applying Lemma 3 to G_1 and comparing (2.2) and (2.3).

Proof of Lemma 3. We use the symmetrization technique (see [B], [BG], [G1], [G2], [PS].) First, one can assume that $\phi \ge 0$: replacing $\phi(x)$ by $|\phi(x)|$ does not result in the change of either the right hand side or the left hand side in (2.3). For $t \ge 0$, let $m_\phi(t)$ be the measure of the set $\{x \in G : \phi(x) < t\}$; this is a lower semi-continuous function that increases from 0 to $M = \max \phi(x)$. One can uniquely define a continuous, non-decreasing function $\phi^*(s)$ on the interval $[0, l(G)]$ such that $\phi^*(0) = 0$ and $m_{\phi^*}(t) = m_\phi(t)$. Then

$$\int_G |\phi(x)|^2 dx = \int_0^M t^2 dm_\phi(t) = \int_0^{l(G)} |\phi^*(s)|^2 ds. \tag{2.4}$$

The set of $H^1_y(G)$ functions that are continuously differentiable on closed edges is dense in $H^1_y(G)$; therefore, for the proof of (2.3), one can assume that $\phi(x)$ is continuously differentiable on closed edges. A critical point of $\phi(x)$ is either a critical point on an open edge or a vertex. By Sard's theorem the set of critical values have measure 0. Let t be a regular value of $\phi(x)$. The number of pre-images of t under $\phi(x)$ is finite; we denote this number by $n(t)$. The co-area formula (e.g., see [B]) implies

$$\int_G |\phi'(x)|^2 dx = \int_0^M dt \sum_{x:\phi(x)=t} |\phi'(x)|.$$

By the Cauchy–Schwarz inequality,

$$\sum_{x:\phi(x)=t} |\phi'(x)| \ge n(t)^2 \Big(\sum_{x:\phi(x)=t} \frac{1}{|\phi'(x)|} \Big)^{-1} \ge \Big(\sum_{x:\phi(x)=t} \frac{1}{|\phi'(x)|} \Big)^{-1} = \frac{1}{m'_\phi(t)}. \tag{2.5}$$

Therefore,

$$\int_G |\phi'(x)|^2 dx \geq \int_0^M \frac{dt}{m'_\phi(t)}. \tag{2.6}$$

The same argument applies to the function $\phi^*(s)$; that function takes every regular value once, and all inequalities become exact equalities. One concludes that

$$\int_G |\phi'(x)|^2 dx \geq \int_0^{l(G)} |(\phi^*)'(s)|^2 ds.$$

Function $\phi^*(s)$ belongs to $H^1([0, l(G)])$ and $\phi^*(0) = 0$. Therefore,

$$\int_0^{l(G)} |(\phi^*)'(s)|^2 ds \geq \frac{\pi^2}{4l(G)^2} \int_0^{l(G)} |\phi^*(s)|^2 ds \tag{2.7}$$

because $\pi/(2l(G))$ is the first eigenvalue of the operator $-d^2/ds^2$ on the interval $[0, l(G)]$, with the Dirichlet condition at 0 and the Neumann condition at $l(G)$.

This finishes the proof of the inequality (2.3). Now, suppose that an equality in (2.3) takes place for a non-zero function $\phi(x)$. Then

(1) the function $\phi(x)$ minimizes the Rayleigh quotient

$$\int_G |\phi'(x)|^2 dx \Big/ \int_G |\phi(x)|^2 dx$$

on the space $H^1_y(G)$;

(2) the equality in (2.5) holds;

(3) the equality in (2.7) holds.

The first condition implies that $\phi(x)$ is an eigenfunction of the Laplacian on G, with the Dirichlet condition at the point y. Therefore, on each edge of $G \setminus y$, it is a trigonometric function. The same is true for $|\phi(x)|$ because, for that function an equality in (2.3) also holds. The second condition implies that $n(t) = 1$ for all regular values t. We conclude that y is a vertex of G of degree 1 (a leaf.) In fact, the derivative of $|\phi(x)|$ at y in each direction emanating from y is positive (it can not vanish), so if there is more than one direction then small positive values are taken at least twice. In the same way, G does not have vertices of degree greater than 2. If v is a vertex of degree at least 3, then, close to v, the function $\phi(x)$ either increases or decreases along each edge; so either $\phi(x)$ or $-\phi(x)$ increases in a neighborhood of v along two different edges emanating from v. Therefore the values that are close to $\phi(v)$ either from above or from below are taken at least twice.

We have agreed to disregard vertices of degree 2. Finally, G is a connected graph, and all its vertices are leaves. There is at least one vertex (y.) Therefore, G is a segment $[0, l(G)]$, and $\phi(x)$ is a monotone function on that segment. That implies $\phi = \phi^*$. The third condition tells us that ϕ^* is the first eigenfunction of the Laplacian on $[0, l(G)]$, with the Dirichlet condition at 0 and the Neumann condition at $l(G)$, so it is proportional to $\sin(\pi s/(2l(G)))$. □

2B. Proof of Lemma 2. The proof of Lemma 2 is based on the following lemma.

Lemma 4. *Let G be a connected metric tree of length L. For every l, $0 < l \le L$, there exists a point $x \in G$ such that*

$$G(x) = G_0 \sqcup G_1 \sqcup \cdots \sqcup G_p,$$

and $l(G_0) \le L - l$, $l(G_k) \le l$, $1 \le k \le p$.

One applies Lemma 4 $(j-1)$ times. Fix $l = L/j$. First, one finds a point x_1 such that $G(x_1) = G_1 \sqcup G^{(1)}$ where G_1 is a connected tree of length $\le (j-1)L/j$, and all connected components of $G^{(1)}$ have length $\le L/j$. Then one finds $x_2 \in G_1$ such that $G_1 = G_2 \sqcup G^{(2)}$, with G_2 being a connected tree of length $\le (j-2)L/j$, and all connected components of $G^{(2)}$ having length $\le L/j$. One keeps going, and, after not more than $(j-1)$ steps, one gets the desired decomposition.

Proof of Lemma 4. We fix a leaf y_0 of G. For a point $x \in G$ that is not a vertex, we denote by G_x the connected component of $G(x)$ that does not contain y_0. Note that, if x is not a vertex, then $G(x)$ consists of exactly two connected components. If $l(G_x) = l$ for some $x \in G \setminus V$ (here V is the set of vertices) then such a point will do the job. Otherwise, on each edge e of G, either $l(G_x) < l$, $x \in e$, (we call them edges of the first type) or $l(G_x) > l$, $x \in e$; they will be called edges of the second type. Denote by G^1 the closure of the union of all edges of the first type; G^2 is the closure of the union of edges of the second type. All connected components of both G^1 and G^2 are metric trees. Notice that the edge incident to y_0 belongs to G^2, and the edges that are incident to all other leaves of G belong to G^1. Let $y \ne y_0$ be a leaf of G_2. By G_0 we denote the component of $G(y)$ that contains y_0, and let $G_1, \ldots G_p$ be other components of $G(y)$.

We claim that $l(G_0) \le L-l$ and $l(G_k) \le l$, $1 \le k \le p$. In fact, let e_k, $0 \le k \le p$, be the edge of G_k incident to y (notice that y is a leaf for all G_js.) For $x \in e_0$, one has $l(G_x) \ge l$, and

$$l(G_0) = \lim_{e_0 \ni x \to y} l(G \setminus G_x) \le L - l.$$

Because y is a leaf of G^2, the edges $e_1, \ldots, e_p$ belong to G^1; therefore, for $1 \le k \le p$, one has

$$l(G_k) = \lim_{e_k \ni x \to y} l(G_x) \le l.$$

□

2C. The case of equality in (1.6). To finish the proof of Theorem 1 we have to analyze, under what conditions the equality in (1.6) takes place. First, we consider the case when G is a connected tree. Then, for any linear combination $\phi(x)$ of $\phi_1(x), \ldots, \phi_j(x)$ that vanishes at the points $x_1, \ldots, x_m$ from Lemma 2, the inequality (2.1) becomes an exact equality. Therefore, $\phi(x)$ is an eigenfunction of the Laplacian on G that corresponds to the eigenvalue $\mu_j(G) = \pi^2 j^2/(4l(G)^2)$. Let $G_1, \ldots, G_p$ be the connected components of $G(x_1, \ldots, x_m)$. The restriction of $\phi(x)$ to G_k, $k = 1, \ldots, p$, if not identically zero, is an eigenfunction of the Laplacian on G_k, with the Dirichlet condition at those points x_i that belong to G_k. From Lemma 3 (notice that the length of each G_k does not exceed $l(G)/j$) we conclude that those components G_k, on which the function $\phi(x)$ does not vanish identically, are segments of length $l(G)/j$, one endpoint of each segment is one of the points $x_1, \ldots, x_m$, and the restriction of $\phi(x)$ to such a segment is proportional to $\sin(\pi j s/(2l(G))$ where s is the distance to the endpoint of the segment where

$\phi(x)$ vanishes. The function $\phi(x)$ does not vanish at the second end of the segment, so the second end of the segment is a leaf of the tree G because this segment is a connected component of $G(x_1, \dots , x_m)$.

A certain complication arises from the fact that $\phi(x)$ may vanish on some of the components G_k: an eigenfunction of the Laplacian on a metric graph may well vanish on some edges of the graph. Now, we do induction in j. If $j = 2$ then $m = 1$, and one has only one point x_1. The function $\phi(x)$ does not vanish on at least two connected components of $G(x_1)$: otherwise $\phi(x)$ would not satisfy the Kirchhoff condition at the point x_1 (notice that x_1 is not a leaf of G; if x_1 is not a vertex then the Kirchhoff condition is the same as the differentiability at x_1 condition.) Each connected component of $G(x_1)$ on which $\phi(x)$ does not vanish is of length $l(G)/2$, so there are exactly two of them, and these are the only connected components of $G(x_1)$. We conclude that G consists of two segments of length $l(G)/2$ emanating from x_1, so G is a segment, and x_1 is its midpoint.

Now, let us do the inductive step. Let $j \geq 3$. Let G_1 be a connected component of $G(x_1, \dots , x_m)$ on which $\phi(x)$ does not vanish. Suppose that x_1 is an endpoint of G_1. As we have already seen, G_1 is a segment of length $l(G)/j$ than connects x_1 with a leaf of the graph G. Therefore, $G' = G \setminus G_1$ is a connected tree, x_1 is one of its vertices, and $l(G') = (j - 1)l(G)/j$. By $\mathcal{L}$ we denote the space of all linear combinations of $\phi_1(x), \dots , \phi_j(x)$ that vanish at x_1. Clearly, $\dim \mathcal{L} = j - 1$. A non-zero function $\psi(x) \in \mathcal{L}$ can not vanish identically on G'. In fact, if it vanishes on G', then

$$\int_{G_1} |\psi'|^2 dx \leq \frac{\pi^2}{4l(G_1)^2} \int_{G_1} |\psi(x)|^2 dx,$$

so the restriction of $\psi(x)$ to G_1 is proportional to $\sin(\pi s/(2l(G_1))$, and the Kirchhoff condition breaks at the point x_1. Denote by $\mathcal{L}'$ the space of restrictions of functions from $\mathcal{L}$ to G'. Then

$$\dim \mathcal{L}' = j - 1. \tag{2.8}$$

For every $\psi \in \mathcal{L}$, one has

$$\int_G |\psi'(x)|^2 dx \leq \frac{\pi^2 j^2}{4l(G)^2} \int_G |\psi(x)|^2 dx$$

and

$$\int_{G_1} |\psi'(x)|^2 dx \geq \frac{\pi^2 j^2}{4l(G)^2} \int_{G_1} |\psi(x)|^2 dx.$$

Therefore,

$$\begin{aligned} \int_{G'} |\psi'(x)|^2 dx &\leq \frac{\pi^2 j^2}{4l(G)^2} \int_{G'} |\psi(x)|^2 dx \\ &= \frac{\pi^2 (j-1)^2}{4l(G')^2} \int_{G'} |\psi(x)|^2 dx. \end{aligned} \tag{2.9}$$

From (2.8) and (2.9), one concludes that

$$\mu_{j-1}(G') \leq \frac{\pi^2 (j-1)^2}{4l(G')^2},$$

and, by the induction assumption, $G' = H_{j-1}$. In the case $j = 3$, we treat a segment as H_2 by inserting a vertex at the midpoint of the segment. Denote by y the center of $G' = H_{j-1}$. The question is, how the segment G_1 is attached to G'. There are three possibilities:

(1) $x_1 = y$;
(2) x_1 lies inside of an edge $e = (y, z)$ of G';
(3) x_1 coincides with a leaf z of G'.

In the first case, $G = H_j$, so we have to rule out two remaining possibilities.

Suppose that x_1 lies inside of (y, z). Let $G'' = G' \setminus (x_1, z]$. Every function $\psi \in \mathcal{L}'$ satisfies

$$\int_{G''} |\psi'(x)|^2 dx \leq \frac{\pi^2(j-1)^2}{4l(G')^2} \int_{G''} |\psi(x)|^2 dx \tag{2.10}$$

because

$$\int_{(x_1,z)} |\psi'(x)|^2 dx \geq \frac{\pi^2(j-1)^2}{4l(G')^2} \int_{(x_1,z)} |\psi(x)|^2 dx \tag{2.11}$$

(notice that the length of (x_1, z) is smaller than $l(G)/j = l(G')/(j-1)$.) A function $\psi \in \mathcal{L}'$ can not vanish on G'' because, otherwise, a strict inequality would hold in (2.11), and that would contradict (2.9). Therefore, the inequality (2.10) holds for functions from a $(j-1)$-dimensional subspace of $H^1(G'')$. Hence,

$$\mu_{j-1}(G'') \leq \frac{\pi^2(j-1)^2}{4l(G')^2} < \frac{\pi^2(j-1)^2}{4l(G'')^2}.$$

The last inequality contradicts (1.6).

Let us now treat the case $x_1 = z$. Then the graph G consists of $(j-2)$ edges, $e_1, \ldots, e_{j-2}$ emanating from y, of length L/j each, and one edge, f, of length $2L/j$ emanating from y (here $L = l(G)$.) All edges connect y with leaves of G. We parametrize each edge by the distance from y. An eigenfunction of the Laplacian on G that corresponds to an eigenvalue $\mu = \lambda^2 \neq 0$ equals $a_k \cos(\lambda((L/j) - s))$ on an edge e_k, and it equals $b \cos(\lambda((2L/j) - s))$ on the edge f. When $s = 0$, all the values must coincide, so

$$a_1 \cos(\lambda L/j) = \cdots = a_{j-2} \cos(\lambda L/j) = b \cos(2\lambda L/j). \tag{2.12}$$

The Kirchhoff condition at y reads

$$(a_1 + \cdots + a_{j-2}) \sin(\lambda L/j) + b \sin(2\lambda L/j) = 0. \tag{2.13}$$

We will count the number of eigenvalues of the Laplacian on G that do not exceed $\pi^2 j^2/(4L^2)$. There is an eigenvalue 0 of multiplicity 1. In the case $\cos(\lambda L/j) = 0$, (2.12) and (2.13) imply $a_1 + \cdots + a_{j-2} = 0$ and $b = 0$; one gets a $(j-3)$-dimensional space of eigenfunctions that correspond to the eigenvalue $\pi^2 j^2/(4L^2)$. If $\sin(\lambda L/j) = 0$ then $\mu = \lambda^2 \geq (\pi^2 j^2/L^2) > \pi^2 j^2/(4L^2)$. In the case when $\cos(\lambda L/j) \neq 0$ and $\sin(\lambda L/j) \neq 0$, (2.12) and (2.13) imply $a_1 = \cdots = a_{j-2} = a$,

$$a \cos(\lambda L/j) = b(2\cos^2(\lambda L/j) - 1), \quad \text{and} \quad \cos^2(\lambda L/j) = \frac{j-2}{2(j-1)}.$$

Therefore, $\cos(2L\lambda/j) = -1/(j-1)$. This case gives rise to one eigenvalue $\arccos^2(-1/(j-1))j^2/(4L^2)$ of multiplicity one that is smaller than $\pi^2 j^2/(4L^2)$; all other eigenvalues are bigger than $\pi^2 j^2/(4L^2)$. Finally, in the case $x_1 = z$, there are exactly $(j-1)$ eigenvalues of G that are smaller than or equal to $\pi^2 j^2/(4l(G)^2)$, so, for such a graph, an equality in (1.6) does not take place.

We have proved that if an equality in (1.6) takes place, and if G is a connected tree, then $G = H_j$. If G is a connected graph that is not a tree then one can cut it at points $x_1, \ldots, x_m$ lying on open edges in such a way that $G' = G(x_1, \ldots, x_m)$ is a connected tree. As it was noted earlier, $\mu_j(G') \le \mu_j(G)$. If $\mu_j(G) = \pi^2 j^2/(4l(G)^2)$ then the last inequality, in combination with (1.6), imply $\mu_j(G') = \pi^2 j^2/(4l(G')^2)$. Therefore, $G' = H_j$ for any choice of points $x_1, \ldots, x_m$ that make $G(x_1, \ldots, x_m)$ a connected tree. Clearly, this is impossible. □

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