Dirichlet kernel, convergence of Fourier series, and Gibbs phenomenon

In these notes we discuss convergence properties of Fourier series. Let $f(x)$ be a periodic function with the period 2π. This choice for the period makes the annoying factors π/L disappear in all formulas. The Fourier series for the function $f(x)$ is

$$a_0 + \sum_{k=0}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)dy, \quad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos(ky)dy, \quad \text{and} \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin(ky)dy.$$

One substitutes the values of a_k and b_k into partial sums of the Fourier series

$$S_n(x) = a_0 + \sum_{k=0}^{n} (a_k \cos(kx) + b_k \sin(kx))$$

to get

$$\begin{aligned} S_n(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[1 + 2 \sum_{k=0}^{n} (\cos(kx)\cos(ky) + \sin(kx)\sin(ky)) \right] f(y)dy \\ &= \int_{-pi}^{\pi} D_n(x-y) f(y)dy \end{aligned}$$

where

$$D_n(z) = \frac{1}{2\pi} \left[1 + 2 \sum_{k=1}^{n} \cos(kz) \right]. \tag{1}$$

Here we have used the formulas $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. We multiply both sides of (1) by $\sin(z/2)$ and use the formula $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$:

$$\sin\left(\frac{z}{2}\right) D_n(z) = \frac{1}{2\pi} \left\{ \sin\left(\frac{z}{2}\right) + \sum_{k=1}^{n} \left[\sin\left(\left(k+\frac{1}{2}\right)z\right) - \sin\left(\left(k-\frac{1}{2}\right)z\right) \right] \right\}.$$

One regongizes a telescopic sum on the right in the last formula; all terms except of

$$\sin\left(\left(n+\frac{1}{2}\right)z\right)$$

cancel. Therefore,

$$D_n(z) = \frac{\sin\left(\left(n+\frac{1}{2}\right)z\right)}{2\pi \sin\left(\frac{z}{2}\right)}. \tag{2}$$

The function $D_n(z)$ is called the Dirichlet kernel; partial sums of the Fourier series are given by the formula

$$S_n(x) = \int_{-\pi}^{\pi} D_n(x-y)f(y)dy. \tag{3}$$

Formula (2) is actually instrumental for the proof of the Fourier theorem. I will sketch the proof.

First, formula (1) implies

$$\int_{-\pi}^{\pi} D_n(z)dz = 1. \tag{4}$$

Suppose that a function $f(x)$ is piecewise smooth. We use (4) to write

$$S_n(x) - f(x) = \int_{-\pi}^{\pi} D_n(x-y)[f(y) - f(x)]dy. \tag{5}$$

Indeed,

$$\int_{-\pi}^{\pi} D_n(x-y)f(x)dy = f(x)\int_{-\pi}^{\pi} D_n(x-y)dy = f(x).$$

Here we used the fact that D_n is a 2π-periodic function. If the function f is differentiable at the point x then the function

$$g(y) = \frac{f(y) - f(x)}{x - y}$$

is piecewise continuous on the interval $[-\pi, \pi]$, and so is the function

$$h(y) = \frac{f(y) - f(x)}{2\pi \sin\left(\frac{x-y}{2}\right)}.$$

Then

$$S_n(x) - f(x) = \int_{-\pi}^{\pi} \sin\left((n + \frac{1}{2})(x-y)\right)h(y)dy.$$

One can prove that the last integrals approach 0 as $n \to \infty$. The reason is that, as n gets larger, the sin function becomes more and more oscillating, and the contributions to the integral from the intervals where it is positive and contributions from the intervals where it is negative almost cancel.

In the case when the function f has a jump at the point x, one can use

$$\int_{0}^{\pi} D_n(z)dz = \int_{-\pi}^{0} D_n(z)dz = \frac{1}{2} \tag{6}$$

and 2π-periodicity of the functin D_n to write

$$\begin{aligned} S_n(x) - \frac{f(x^+) + f(x^-)}{2} &= \frac{1}{2}\int_{x-\pi}^{x} D_n(x-y)[f(y) - f(x^-)]dy \\ &+ \frac{1}{2}\int_{x}^{x+\pi} D_n(x-y)[f(y) - f(x^+)]dy. \end{aligned}$$

After that, the argument is similar to the argument in the case when x is a point of differentiability of the function f.

Let us now study the partial sum of the Fourier series for the step function $h(x)$; it equals 1 when $0 < x < \pi$ and it equals 0 when $-\pi < x < 0$; then it extended to be 2π-periodic. let x be a small positive number. Then

$$S_n(x) = \int_0^{\pi} D_n(x-y)dy = \int_{x-\pi}^{x} D_n(z)dz.$$

We use (6) to write

$$S_n(x) = \frac{1}{2} + \int_0^{x} D_n(z)dz - \int_{-\pi}^{-\pi+x} D_n(z)dz. \tag{7}$$

Let us fix a number $\xi > 0$, and take a sequence of points $x_n = \xi/(n+(1/2))$. We notice that $D_n(-\pi) = (-1)^n/(2\pi)$, so, for small values of x, $|D_n(z)| < (1/2)$ when $-\pi \le z \le -\pi + x$. Therefore

$$\left| \int_{-\pi}^{-\pi+x_n} D_n(z)dz \right| \le \frac{x_n}{2} < \frac{\xi}{2n}. \tag{8}$$

Let

$$\tilde{D}_n(z) = \frac{\sin\left(\left(n+\frac{1}{2}\right)z\right)}{\pi z}.$$

One checks that the function

$$\frac{1}{\sin\left(\frac{z}{2}\right)} - \frac{2}{z}$$

is continuous and differentiable at the point 0; moreover, it vanishes at the point 0. Therefore, for small values of x, $|D_n(z) - \tilde{D}_n(z)| < (1/2)$ when $0 \le z \le x$, and

$$\left| \int_0^{x_n} D_n(z)dz - \int_0^{x_n} \tilde{D}_n(z)dz \right| < \frac{x_n}{2} < \frac{\xi}{2n}. \tag{9}$$

In the integral of $\tilde{D}_n(z)$, we make a substitution $w = (n+(1/2))z$:

$$\int_0^{x_n} \tilde{D}_n(z)dz = \int_0^{\xi} \frac{\sin w}{\pi w} dw = \frac{\mathrm{Si}(\xi)}{\pi} \tag{10}$$

where

$$\mathrm{Si}(\xi) = \int_0^{\xi} \frac{\sin w}{w} dw$$

is called the Integral Sine function. Its maximal value is assumed when $\xi = \pi$, and $\mathrm{Si}(\pi) \approx 1.85194$. It follows from (7)–(10) that if one takes a sequence of numbers

$$x_n = \frac{2\pi}{2n+1}$$

then

$$\lim_{n\to\infty} S_n(x_n) = \frac{1}{2} + \frac{\mathrm{Si}(\xi)}{\pi} \approx 1.08949.$$

We see that, at the points x_n that are closer and closer to 0 as $n \to \infty$, the partial sums of the Fourier series overshoot the maximal value of $h(x)$ by about 9 percent. It turns out that this is always the case when a function has a jump. This is what is called the Gibbs phenomenon.