BANACH–TARSKI THEOREM

The Banach-Tarski Theorem says that one can divide a ball in the three-dimensional Euclidean space into a finite number of pieces that can be re-arranged to make two balls of the same radius. In particular, it shows that one can not constuct a non-trivial finitely additive translationally invariant measure in $\mathbb{R}^3$. The proof that I give in these notes is an adaptation of the proof from [1]. The book [1] contains a lot of interesting things that are related to the Banach–Tarski Theorem, and it also contains some fun stuff. Let me give one example. Take several copies of a regular tetrahedron of, say, side 1. Put them in a sequence in such a way that two consecutive tetrahedra share exactly one face and every tetrahedron is different from its predecessor's predecessor; you will get a snake. Question: Is it possible to make a snake in such a way that the last tetrahedron is a translation of the first one? This problem was posed by Steinhaus. You can try to find a solution (this is rather difficult a problem.) Then you may look at Theorem 5.10 in [1] (you do not have to read the rest of the book to understand the proof.)

First, we have to say what the words "can be re-arranged" mean exactly. A mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ is called an *isometry* if $|G(x) - G(y)| = |x - y|$ for any choice of $x, y \in \mathbb{R}^n$. In words, G is an isometry if it preserves distances between points. Let us give examples of isometries.

Example 1. Let $b \in \mathbb{R}^n$ be a fixed vector, and let $G(x) = x + b$ be a translation by the vector b. Obviously, it is an isometry.

Example 2. Let A be an $n \times n$-matrix, and let $G(x) = Ax$ be the corresponding linear transformation of $\mathbb{R}^n$. It is an isometry if and only if A is an orthogonal matrix, i.e. $A^T A = I$ where A^T is the transposed of A. The set of all orthogonal matrices is denoted by $O(n)$. An example of an orthogonal transformation in $\mathbb{R}^3$ is a rotation about an axis passing through the origin. Another example is a reflection about a plane passing through the origin.

One can combine examples 1 and 2: any mapping

$$G(x) = Ax + b, \quad A \in O(n), b \in \mathbb{R}^3 \tag{1}$$

is an isometry. The following problem says that these are all isometries.

Problem 1. Prove that every isometry in $\mathbb{R}^n$ is given by formula (1).

Hint. First, reduce the problem to the case when $G(0) = 0$. If $G(0) = 0$ then prove that G is a linear mapping: $G(cx) = cG(x)$, $G(x + y) = G(x) + G(y)$ where $x, y \in \mathbb{R}^n$, $c \in \mathbb{R}$.

Isometries of $\mathbb{R}^n$ are also called *rigid motions*, and the set of all rigid motions is usually denoted by $E(n)$. Rigid motions form a group. Let $G_j(x) = A_j x + b_j$, $j = 1, 2$. Then $G_1 G_2(x) = A_1 A_2 x + A_1 b_2 + b_1$. The determinant of an orthogonal matrix equals either 1 or -1. The set of all orthogonal matrices, the determinant of which equals 1, form a subgroup of $O(n)$; this subgroup is denoted by $SO(n)$. Rigid motions (1) with $\det A = 1$ are called directed rigid motions, and they form

Typeset by $\mathcal{AMS}$-TEX

a subgroup in $E(n)$; this subgroup is denoted by $E^+(n)$.
Problem 2. Let $A \in SO(3)$.
a) Show that $G(x) = Ax$ is a rotation about a line in $\mathbb{R}^3$ that passes through the origin.
b) Show that either the set of fixed points of $G(x)$ on the unit sphere, $\{x \in \mathbb{R}^3 : |x| = 1, Ax = x\}$, consists of exactly two points or $A = I$.
Now, I am ready to formulate the Banach–Tarski Theorem exactly. Recall that a collection of subsets S_α of a set S is called a partition of S if $\cup S_\alpha = S$ and $S_\alpha \cap S_\beta = \emptyset$ when $\alpha \neq \beta$.

Definition 1. Let S be a set in $\mathbb{R}^n$. We will say that it has doubling property if there exist numbers $k < m$, a finite partition $\{S_1, \dots , S_m\}$ of S, and rigid motions $G_j(x) = A_j x + b_j$, $A_j \in SO(n)$, $b_j \in \mathbb{R}^n$, such that $\{G_1(S_1), \dots G_k(S_k)\}$ is a partition of S and $\{G_{k+1}(S_{k+1}), \dots , G_m(S_m)\}$ is also a partition of S.

Let S be a bounded set that has doubling property, and let $b \in \mathbb{R}^n$ be a vector such that $|b|$ is bigger than the diameter of S. Let

$$\tilde{G}_j(x) = \begin{cases} G_j(x), & \text{if } j \leq k \\ G_j(x) + b, & \text{if } j > k. \end{cases}$$

Then the sets $\tilde{G}_j(S_j)$, $1 \leq j \leq m$ form a partition of the union of two non-intersecting sets; one of them is S, and another one is a translation of S. One can say that the set S can be subdivided into a finite number of pieces that can be re-arranged to make two copies of S.

Theorem (Banach–Tarski). *The unit ball in $\mathbb{R}^3$, $B = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ has doubling property.*

The proof of the Banach–Tarski Theorem is based on elementary group theory. A free group of rank k consists of the unity e and the collection of finite words using letters $\sigma_1, \dots , \sigma_k, \sigma_1^{-1}, \dots , \sigma_k^{-1}$, with the only cancellation rules $\sigma_j \sigma_j^{-1} = \sigma_j^{-1} \sigma_j = e$. The product of two words w_1 and w_2 is their concatination; the word w_2 is placed on the right. A word that does not contain combinations $\sigma_j \sigma_j^{-1}$ or $\sigma_j^{-1} \sigma_j$ is called a reduced word. Every element of F_k has a unique representation as a reduced word. The first ingredient of the proof is given by the following proposition

Proposition 1. *The group $SO(3)$ contains a subgroup that is isomorphic to F_2.*

Proof. Let σ and τ be counterclockwise rotation through the angle arccos(1/3) around the z-axis and x-axis, respectively. Then

$$\sigma^{\pm 1} = \begin{pmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau^{\pm 1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}. \tag{2}$$

Let F be the subgroup of $SO(3)$ that is generated by σ and τ. Every element of F can be represented by a word

$$w = \alpha_k \cdots \alpha_1 \tag{3}$$

where $\alpha_j = \sigma^{\pm 1}$ or $\alpha_j = \tau^{\pm 1}$. To show that F is a free group of rank 2 generated by σ and τ one has to check that $w \neq e$ if w is a reduced word of positive length.

Recall that a reduced word is such a word that σ is never adjacent to σ^{-1} and τ is never adjacent to τ^{-1}. Let (3) be a reduced word. Suppose that $\alpha_1 = \sigma^{\pm 1}$ (the case $\alpha_1 = \tau^{\pm 1}$ is similar.) Let $m \le k$ and $w_m = \alpha_m \cdots \alpha_1$. I claim that

$$w_m \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 3^{-m} \begin{pmatrix} a_m \\ b_m\sqrt{2} \\ c_m \end{pmatrix} \tag{4}$$

where a_m, b_m, and c_m are integer numbers; moreover, b_m is not divisible by 3. We prove (4) by induction. First,

$$\sigma^{\pm 1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ \pm 2\sqrt{2} \\ 0 \end{pmatrix},$$

so (4) holds for $m = 1$ with $a_1 = 1$, $b_1 = \pm 2$, and $c_1 = 0$. Secondly, a simple computation shows that if the representation (4) holds for w_m then it also holds for w_{m+1} with

$$a_{m+1} = a_m \mp 4b_m, \quad b_{m+1} = \pm 2a_m + b_m, \quad c_{m+1} = 3c_m$$

if $\alpha_{m+1} = \sigma^{\pm 1}$ or

$$a_{m+1} = 3a_m, \quad b_{m+1} = b_m \mp 2c_m, \quad c_{m+1} = \pm 4b_m + c_m$$

if $\alpha_{m+1} = \tau^{\pm 1}$. To show that b_k is not divisible by 3 we will reduce integers a_j, b_j, and c_j mod 3. Introduce vectors $x_m = (a_m, b_m, c_m)^T \pmod 3$. Then

$$x_{m+1} = \begin{pmatrix} 1 & \mp 1 & 0 \\ \mp 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_m = A_\pm x_m, \quad \text{if} \quad \alpha_{m+1} = \sigma^{\pm 1}$$

or

$$x_{m+1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \pm 1 \\ 0 & \pm 1 & 1 \end{pmatrix} x_m = B_\pm x_m, \quad \text{if} \quad \alpha_{m+1} = \tau^{\pm 1}.$$

The word w can be represented as

$$w = \tau^{p_l} \sigma^{p_{l-1}} \tau^{p_{l-2}} \cdots \sigma^{p_1}$$

where p_j are integer numbers and $p_j \ne 0$ if $j \le l-1$. Then

$$x_k = B_\pm^{|p_l|} A_\pm^{|p_{l-1}|} \cdots A_\pm^{|p_1|} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Notice that matrices $A_\pm$ and $B_\pm$ have rank 1 and their non-zero eigenvalue equals $2 = -1 \pmod 3$. Therefore

$$A_\pm^n = (-1)^n A_\pm \pmod 3 \quad \text{and} \quad B_\pm^n = (-1)^n B_\pm \pmod 3;$$

here n is a positive, integer number. We conclude that

$$x_k = (-1)^k B_\pm A_\pm \cdots B_\pm A_\pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad x_k = (-1)^k A_\pm B_\pm \cdots B_\pm A_\pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

I claim that

$$A_\pm B_\pm \cdots B_\pm A_\pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \tag{5}$$

and

$$B_\pm A_\pm \cdots B_\pm A_\pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ c \end{pmatrix} \tag{6}$$

where $b \neq 0$. This fact can be proved by induction in the length of the word. Firstly,

$$A \pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \mp 1 \\ 0 \end{pmatrix}.$$

If (5) holds, and one applies $B_\pm$ to both sides of (5) then one gets

$$B_\pm \cdots A_\pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = B_\pm \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ \pm b \end{pmatrix}.$$

If (6) holds, and one applies $A_\pm$ to both sides of (6) then one gets

$$A_\pm \cdots A_\pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = A_\pm \begin{pmatrix} 0 \\ b \\ c \end{pmatrix} = \begin{pmatrix} \mp b \\ b \\ 0 \end{pmatrix}.$$

Actually, the value of the second component of the vector $(a, b, c)^T$ does not change, so $b_k = \pm(-1)^k \pmod 3$. In particular,

$$w \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

and, therefore, $w \neq I$.

□

The next Proposition says, roughly speaking, that one can partition a free group of rank 2 in four pieces that can be rearranged to make two groups of the same size. To formulate the Proposition, let me introduce some notations. For a subset S of a group Γ and for $x \in \Gamma$, we write $xS = \{xy : y \in S\}$.

Proposition 2. *There exists a partition $\{G_1, G_2, G_3, G_4\}$ of a free group F_2 generated by σ and τ such that $\tau G_2 = F_2 \setminus G_1$ and $\sigma G_4 = F_2 \setminus G_3$.*

Proposition 2 implies $F_2 = G_1 \cup \tau G_2 = G_3 \cup \sigma G_4$; in other words, the group F_2 can be obtained by rearranging G_1 and G_2, and it can be also obtained by rearranging G_3 and G_4.

Proof. Let w be any reduced word written in the alphabet $\sigma^{\pm 1}, \tau^{\pm 1}$. By $W(w)$ we denote the set of all reduced words that start with w (on the left.) Define

$$G_1 = W(\tau), \quad G_2 = W(\tau^{-1}),$$
$$G_3 = W(\sigma) \cup \{e, \sigma^{-1}, \sigma^{-2}, \dots\}, \quad G_4 = W(\sigma^{-1}) \setminus \cup_{k=1}^{\infty} \sigma^{-k}.$$

One has $W(\tau^{-1}) = \{\tau^{-1} w\}$ where w is any word that does not start with the letter τ (in particular, it can be e.) In other words, $G_2 = W(\tau^{-1}) = \tau^{-1}(F_2 \setminus G_1)$. Therefore, $\tau G_2 = F_2 \setminus G_1$. Similarly, $\sigma W(\sigma^{-1}) = F_2 \setminus W(\sigma)$. Therefore,

$$\sigma G_4 = \sigma(W(\sigma^{-1}) \setminus \{e, \sigma^{-1}, \sigma^{-2}, \dots\}) = F_2 \setminus (W(\sigma) \cup \cup_{k=1}^{\infty} \sigma^{-k}) = F_2 \setminus G_3.$$

□

Now, I will briefly discuss group actions. Let Γ be a group, and let X be an arbitrary set. A Γ-action on X is a correspondence $\Gamma \ni \gamma \mapsto T_\gamma$ where $T_\gamma : X \to X$ is a mapping of X into itself such that $T_{\gamma_1 \gamma_2}(x) = T_{\gamma_1}(T_{\gamma_2}(x))$ for every $\gamma_1, \gamma_2 \in \Gamma$ and $x \in X$ and T_e is the identity mapping.

Remark. Actually, the above definition is the definition of a right action. In the literature, you may (and will) see the definition with $T_{\gamma_1 \gamma_2}(x) = T_{\gamma_2}(T_{\gamma_1}(x))$; that will be the definition of a left action.

It follows from the definition that all the maps T_γ are bijections ($T_{\gamma^{-1}}$ is both the left inverse and the right inverse to T_γ.) I will use the common convension to drop the letter T and to write γx for $T_\gamma x$. Let $x \in X$. Then the *orbit* of the point x is the set $O(x) = \{\gamma x : \gamma \in \Gamma\}$.

Problem 3. a) Prove that $O(x) = O(y)$ if and only if $y \in O(x)$.
b) Let $\Gamma = SO(3)$ and $X = \mathbb{R}^3$. The action of Γ on X is given by applying a matrix to a vector. Describe geometrically all orbits of this action.
c) The same question as b), with $SO(3)$ replaced by $GL(3)$, the group of all real-valued 3×3-matrices, the determinant of which is different from 0.
You can see from Problem 3(b,c) that different orbits may look differently. However, in some cases all the orbits look the same. An action of Γ on X is called a *free action* if $\gamma x \neq x$ whenever $\gamma \neq e$ and $x \in X$.

Fact. *Every orbit of a free action of a group Γ on a set X is in a one-to-one correspondence with the group Γ.*

In fact, let $O(x)$ be an orbit of the action. One can define a mapping $\Gamma \to O(x)$ by $\gamma \mapsto \gamma x$. This mapping is surjective by the definition of an orbit. It is injective because $\gamma_1 x = \gamma_2 x$ implies $\gamma_2^{-1} \gamma_1 x = x$, and, therefore, $\gamma_2^{-1} \gamma_1 = e$ (the action is free.) This correspondence $\Gamma \to O(x)$ is not canonical: it depends upon the choice of a point in the orbit.

The unit sphere $S = \{x \in \mathbb{R}^3 : |x| = 1\}$ is an orbit of the $SO(3)$-action (see Problem 3(b).) In Proposition 1, we constructed a subgroup F of $SO(3)$ that is isomorphic to F_2; this group is generated by rotations σ and τ (see (2).) The subgroup F acts on S. The following proposition is a version of a theorem that is due to Hausdorff.

Proposition 3. *There exist a countable set $D \subset S$ and a partition $\{A_1, A_2, A_3, A_4\}$ of $S \setminus D$ such that $\tau A_2 = A_2 \cup A_3 \cup A_4$ and $\sigma A_4 = A_1 \cup A_2 \cup A_4$ where σ and τ are given by (2).*

The statement of Proposition 3 is already rather close to the statement of the Banach–Tarski Theorem. It deals with a sphere, not a ball. It says that one can throw away a countable set from a sphere (and a countable set is a tiny subset!) in such a way that the difference, $S \setminus D$ can be partitioned in four pieced. Moreover, $S \setminus D = A_1 \cup \tau A_2 = A_3 \cup \sigma A_4$. Keep in mind that both σ and τ are rigid motions. In particular, $S \setminus D$ has doubling property.

Proof. The group F is countable: there exists only a finite number of words of fixed length, so the set of all words of finite length is countable. Each rotation $\gamma \in F$ that is different from the identity has exactly two fixed points on S (Problem 2(b).) Let us collect all these fixed points together to form a set $Y \subset S$. The set Y is countable. Let $D = \cup_{y \in Y} O(y)$ where $O(y)$ is the orbit of the F-action on S. The cardinality of $O(y)$ can not exceed the cardinality of F (which is countable.) Therefore the set D is countable. The set $X = S \setminus D$ is invariant under the action of F. Moreover, the action of F on X is free. Let O be the set of all orbits of the F-action on X. For every orbit $o \in O$, we pick a pont $x_o \in O$ (here the axiom of choice is used in a significant way!) This choice gives rise to a mapping $F_o : F \to o$: $F_o(\gamma) = \gamma x_o$. Let $\{G_1, G_2, G_3, G_4\}$ be the partition of $F \equiv F_2$ that was constructed in Proposition 2. One sets $A_j = \cup_{o \in O} F_o(G_j)$.

□

It is useful to introduce the following definition.

Definition 2. Two sets S and $\tilde{S}$ in $\mathbb{R}^n$ will be called equidecomposable if there exists a partition $\{M_1, \dots, M_p\}$ of S, a partition $\{\tilde{M}_1, \dots, \tilde{M}_p\}$ of $\tilde{S}$, and rigid motions $H_j(x) = A_j x + b_j$, $A_j \in SO(n)$, $b_j \in \mathbb{R}^n$, $j = 1, \dots, p$, such that $\tilde{M}_j = H_j(M_j)$.

Proposition 4.. *Let S and $\tilde{S}$ be equidecomposable sets in $\mathbb{R}^n$. Suppose that S has doubling property. Then $\tilde{S}$ also has doubling property.*

Proof. In the proof, I will use notations from Definitions 1 and 2. Let $i, j = 1, \dots, p$ and $l = 1, \dots m$. Let

$$\tilde{S}_{ijl} = \{x \in \tilde{M}_j : H_j^{-1}(x) \in S_l, G_l H_j^{-1}(x) \in M_i\}.$$

The sets $\tilde{S}_{ijl}$ form a partition of $\tilde{S}$. Let $\tilde{G}_{ijl} = H_i G_l H_j^{-1}$. I claim that

$$\{\tilde{G}_{ijl}(\tilde{S}_{ijl}), 1 \le i, j \le p, 1 \le l \le m\} \tag{7}$$

is a partition of $\tilde{S}$ and

$$\{\tilde{G}_{ijl}(\tilde{S}_{ijl}), 1 \le i, j \le p, m+1 \le l \le k\} \tag{8}$$

is also a partition of $\tilde{S}$. Let us deal with the sets (7). The fact that they are mutually disjoint is left as an exercise. Let us combine the maps H_j into one map $H : S \to \tilde{S}$: $H(x) = H_j(x)$ when $x \in M_j$. Then

$$\cup_{j=1}^{p} H_j^{-1}(\tilde{S}_{ijl}) = \{x \in S_l : G_l(x) \in M_i\}.$$

The sets $\{G_l(S_l), 1 \leq l \leq k\}$ form a partition of S, so

$$\cup_{l=1}^{k} \cup_{j=1}^{p} G_l H_j^{-1}(\tilde{S}_{ijl}) = M_i,$$

and

$$\cup_{i=1}^{p} \cup_{l=1}^{k} \cup_{j=1}^{p} H_i G_l H_j^{-1}(\tilde{S}_{ijl}) = \tilde{S}.$$

□

Remark. Notice that if all maps G_l in Definition 1 and all maps H_j in definition 2 are linear (all the b's vanish) then the maps $\tilde{G}_{ijl}$ are also linear.

Proposition 3 tells us that the unit sphere, with a countable set D removed has doubling property. The next proposition implies that the unit sphere itself has doubling property.

Proposition 5. *Let D be a countable subset of the unit sphere S in $\mathbb{R}^3$. Then S and $S \setminus D$ are equidecomposable.*

Proof. Choose a line that passes through the origin, and that intesects S in the points that do not belong to D. By ρ_θ I denote the counterclockwise rotation throught the angle θ around this line. For a given point $x \in D$ and given $n > 0$ the set $S_{n,x} = \{\theta : \rho_{n\theta}(x) \in D\}$ is countable. Then the set

$$S = \cup_{x \in D} \cup_{n=1}^{\infty} S_{n,x}$$

is countable. Let $\theta \notin S$, and let $\rho = \rho_\theta$. Then $\rho^n(D) \cap D = \emptyset$ for every positive, integer n. Therefore $\rho^m(D) \cap \rho^n(D) = \emptyset$ if $m \neq n$ ($\rho^m(D) \cap \rho^n(D) = \rho^m(D \cap \rho^{n-m}(D))$ if $m < n$.) Let

$$\bar{D} = D \cup \rho(D) \cup \rho^2(D) \cup \cdots .$$

Clearly, $\rho(\bar{D}) = \bar{D} \setminus D$ and

$$S \setminus D = (S \setminus \bar{D}) \cup \rho(\bar{D}).$$

On the other hand,

$$S = (S \setminus \bar{D}) \cup \bar{D}.$$

□

Corrolary. *The unit ball in $\mathbb{R}^3$ with the center removed, $B' = \{x \in \mathbb{R}^3 : 0 < |x| \leq 1\}$ has doubling property.*

Proposition 5 tells us that there exist a partition $\{S_j, 1 \leq j \leq m\}$ of the unit sphere and rigid motions G_j such that $\{G_j(S_j), 1 \leq j \leq k\}$ form a partition of S and $\{G_j(S_j), k+1 \leq j \leq m\}$ also form a partition of S. Here $k < m$. It follows from the remark after the proof of Proposition 4 that one can assume all G_j's to be linear maps, $G_j(x) = A_j x$, $A \in SO(3)$. Then one takes

$$\bar{S}_j = \left\{ x \in B' : \frac{x}{|x|} \in S_j \right\}$$

and $\bar{G}_j = G_j$.

The following proposition finishes the proof of the Banach–Tarski Theorem.

Proposition 6. *The unit ball B in $\mathbb{R}^3$ and the set $B' = B \setminus \{0\}$ are equidecomposable.*

Proof. Let l be the line $x = 1/2, y = 0$ in $\mathbb{R}^3$ and let ρ be a counterclockwise rotation around l through an angle θ such that θ/π is an irrational number. Denote the origin $(0, 0, 0)$ by P. Notice that all the points $\rho^k(P)$ are different. Let $M = \{\rho^k(P), k \geq 0\}$, and let $M' = \{\rho^k(P), k > 0\} = M \setminus \{P\}$. Then

$$B = (B \setminus M) \cup M \quad \text{and} \quad B' = (B \setminus M) \cup M'$$

are decompositions of B and B';

$$\text{id} : B \setminus M \to B \setminus M \quad \text{and} \quad \rho : M \to M'$$

are the corresponding rigid motions.

□

References

1. S. Wagon, *The Banach–Tarski Paradox*, Cambridge University Press, 1985.
2. K. Stomberg, *The Banach–Tarski Paradox*, American Mathematical Monthly **86** (1979), 151–161.