PROBLEM SET 11

Problem 1

Let M be a compact metric space, and let $0 < \alpha < 1$ Prove that the set of all continuous functions $f(x)$ on M such that $||f|| \leq 1$ ans

$$\sup_{x,y \in M; x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^{\alpha}} \leq 1$$

is compact in $C(M)$. Here $|| \cdot ||$ is the norm in $C(M)$ and $d(x,y)$ is the distance between x and y.

Problem 2

Let $K(x,y) \in C([0,1] \times [0,1])$. Prove that the set of all functions $f(x)$ on $[0,1]$ that can be represented as

$$f(x) = \int_0^1 K(x,y) u(y) dy$$

with $u(x) \in L^1([0,1])$ and

$$\int_0^1 |u(x)| dx \leq 1$$

is precompact in $C([0,1])$.

Problem 3

Let $f_n(z)$ be a sequence of holomorphic functions in the unit disc $\{z : |z| < 1\}$, and $|f_n(z)| \leq 1$ for all n and for all z, $|z| < 1$.
a) Prove that one can find a subsequence that converges uniformly in $\{z : |z| \leq 1/2\}$.
b) Prove that one can find a subsequence that converges pointwise to a holomorphic function in the whole unit disc.

Problem 4

Let M and N be compact metric spaces. Prove that finite linear combinations of functions of the type $f(x)g(y)$ with $f \in C(M)$ and $g \in C(N)$ are dense in $C(M \times N)$.

Problem 5

Prove that finite linear combinations of the functions $\sin(kx)$ with positive, integer k are dense in $C([\epsilon, \pi - \epsilon])$ for every $\epsilon > 0$. Are they dense in $C([0, \pi])$?

Problem 6

Let M be a compact metric space, and let $\mathcal{A}$ be a unital subalgebra od $C(M)$. One does not assume that it separates points. We say that two points $x, y \in M$ are equivalent, $x \sim y$, if $f(x) = f(y)$ for every $f \in \mathcal{A}$. Prove that the closure of $\mathcal{A}$ in $C(M)$ is the set of all continuous functions such that $f(x) = f(y)$ for $x \sim y$.
Hint. For each $x \in M$ define $F_x = \{y \in M : x \sim y\}$. Show that the sets F_x are closed, define a metric space the elements of which are these sets F_x, and prove that it is compact.