PROBLEM SET 5

Problem 1

A real-valued continuous function $f(x)$ on $\mathbb{R}$ is called Hölder at a point $x \in M$ if there exist constants C, $\epsilon > 0$, and α, $0 < \alpha \le 1$, such that

$$|f(y) - f(x)| \le C|x-y|^\alpha, \quad \text{when} \quad |x-y| < \epsilon.$$

Prove that the set of points where a continuous function is Hölder is a Borel set. To which $G_{\dots}$ or $F_{\dots}$ class does it belong?

Problem 2

Let $f(x)$ be a bounded function on an interval $[a,b]$, $-\infty < a < b < \infty$. Let $S = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of the interval $[a,b]$, and let

$$R_S(f) = \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1})$$

be a Riemann sum: here $x_j^* \in [x_{j-1}, x_j]$. Recall that a function $f(x)$ is called Riemann integrable on $[a,b]$ if the limit $\lim_{m(S)\to 0} R_S(f)$ exists; here $m(S) = \max\{x_j - x_{j-1}\}$.

Let $\delta > 0$ be a positive number. For every point $x \in [a,b]$ we introduce a number

$$\operatorname{osc}(f; x, \delta) = \sup_{y\in[a,b]\cap[x-\delta,x+\delta]} f(y) - \inf_{y\in[a,b]\cap[x-\delta,x+\delta]} f(y).$$

a) Prove that the limit

$$\operatorname{osc}(f;x) = \lim_{\delta\to 0} \operatorname{osc}(f;x,\delta)$$

exists.
b) Prove that a function f is continuous at a point x if and only if $\operatorname{osc}(f;x) = 0$.
c) Define $D_\epsilon(f) = \{x : \operatorname{osc}(f;x) \ge \epsilon\}$; here ϵ is a positive number. Prove that if the function $f(x)$ is Riemann integrable then $m(D_\epsilon(f)) = 0$.
d) Let $D(f)$ be the set of all points where the function $f(x)$ is discontinuous. Prove that if the function $f(x)$ is Riemann integrable then $m(D(f)) = 0$.
e) Prove that the sets $D_\epsilon(f)$ are closed.
f) Suppose that $m(D(f)) = 0$. Prove that for every $\epsilon > 0$ and for every $\eta > 0$ there exist a *finite* set of mutually disjoint intervals $[a_j, b_j] \subset [a,b]$, $j = 1, \dots, N$ such that

$$[a,b] \setminus D_\epsilon(f) \supset \cup_{j=1}^N [a_j, b_j] \quad \text{and} \quad \sum_{j=1}^{N} (b_j - a_j) \ge b - a - \eta.$$

g) Let $f(x)$ be a bounded function defined on $[a,b]$. Prove that if $m(D(f)) = 0$ then the function $f(x)$ is Riemann integrable.

Exercises 6, 9, 15. p.p. 91, 92