FINAL EXAM FOR MATH 527B

Problem 1

Let $(X, \mathcal{B})$ be a measurable space, and let $\{f_n(x)\}$ be a sequence of measurable functions on X. Prove that the set of all x for which $\lim_{n\to\infty} f_n(x)$ exists is a measurable set.

Problem 2

Let $(X, \mathcal{B}, \mu)$ be a measure space. One says that a sequence of measurable functions $\{f_n(x)\}$ converges to a function $f(x)$ in measure if

$$\lim_{n\to\infty} \mu\{x : |f_n(x) - f(x)| > \epsilon\} = 0$$

for evey $\epsilon > 0$. Assume that $\mu(X) < \infty$. Prove that a sequence $\{f_n(x)\}$ converges to $f(x)$ in measure if and only if

$$\lim_{n\to\infty} \int_X \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} d\mu = 0.$$

Problem 3

Find the limit

$$\lim_{n\to\infty} \int_0^\infty \frac{n \sin(x/n)}{x(1+x^2)} dx.$$

Justify all steps.

Problem 4

Find all values of $\alpha > 0$ for which the Lesbegue integral

$$\int_{[0,1]\times[0,1]} \frac{x^2 - y^2}{(x^2+y^2)^\alpha} dm$$

exists? Here m is the Lesbegue measure. Justify your answer.

Problem 5

Let $0 < \alpha \leq 1$. A function $f(x)$ is called Hölder continuous of order α if there exists a constant C such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

for all values x and y. Let $f(x)$ be a Hölder continuous function of order α defined on $[0, 1]$, and let

$$\Gamma_f = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, y = f(x)\}$$

be the graph of $f(x)$. Prove that the Hausdorff dimension of Γ_f does not exceed $2 - \alpha$.

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