FINAL EXAM FOR MATH 527B
SOLUTIONS

Problem 1

Let $(X, \mathcal{B})$ be a measurable space, and let $\{f_n(x)\}$ be a sequence of measurable functions on X. Prove that the set of all x for which $\lim_{n\to\infty} f_n(x)$ exists is a measurable set.

Solution

Let S be the set of all x for which $\lim_{n\to\infty} f_n(x)$ exists. A sequence of real numbers converges if and only if it is a Cauchy sequence, so

$$S = \{x : (\forall \epsilon > 0)(\exists N)(\forall n > N)(\forall m > N)\ |f_n(x) - f_m(x)| < \epsilon\}.$$

Let

$$A_{k,m,n} = \left\{ x : |f_n(x) - f_m(x)| < \frac{1}{k} \right\}.$$

The sets $A_{k,m,n}$ are measurable, so the set

$$S = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N+1}^{\infty} \bigcap_{m=N+1}^{\infty} A_{k,m,n}$$

is also measurable.

Problem 2

Let $(X, \mathcal{B}, \mu)$ be a measure space. One says that a sequence of measurable functions $\{f_n(x)\}$ converges to a function $f(x)$ in measure if

$$\lim_{n\to\infty} \mu\{x : |f_n(x) - f(x)| > \epsilon\} = 0$$

for every $\epsilon > 0$. Assume that $\mu(X) < \infty$. Prove that a sequence $\{f_n(x)\}$ converges to $f(x)$ in measure if and only if

$$\lim_{n\to\infty} \int_X \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} d\mu = 0. \tag{1}$$

Solution

Let $S_{n,\epsilon} = \{x : |f_n(x) - f(x)| > \epsilon\}$. Notice that

$$\int_X \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} d\mu \leq \mu(S_{n,\epsilon}) + \epsilon\mu(X).$$

Here we used the fact that

$$\frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \leq \min\{1, |f_n(x) - f(x)|\}.$$

Suppose that f_n converges to f in measure. Given $\delta > 0$, take $\epsilon = \delta/(2\mu(X))$. Then $\mu(S_{n,\epsilon}) < \delta/2$ for sufficiently large n, and

$$\int_X \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} d\mu < \delta.$$

This proves that convergence in measure implies (1). To prove that (1) implies convergence in measure we notice that

$$\mu(S_{n,\epsilon}) \leq \frac{1+\epsilon}{\epsilon} \int_X \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} d\mu.$$

Problem 3

Find the limit

$$\lim_{n\to\infty} \int_0^\infty \frac{n \sin(x/n)}{x(1+x^2)} dx.$$

Justify all steps.

Solution

One has

$$\lim_{n\to\infty} \frac{n \sin(x/n)}{x(1+x^2)} = \frac{1}{1+x^2}$$

for every $x > 0$, and

$$\left| \frac{n \sin(x/n)}{x(1+x^2)} \right| \leq \frac{1}{1+x^2},$$

so

$$\lim_{n\to\infty} \int_0^\infty \frac{n \sin(x/n)}{x(1+x^2)} dx = \int_0^\infty \frac{1}{1+x^2} = \pi/2$$

by the Dominated Convergence Theorem.

Problem 4

Find all values of $\alpha > 0$ for which the Lesbegue integral

$$\int_{[0,1]\times[0,1]} \frac{x^2 - y^2}{(x^2+y^2)^\alpha} dm \tag{2}$$

exists? Here m is the Lesbegue measure. Justify your answer.

Solution

Notice that the integrals of the positive and the negative part of the integrand in (2) are equal to

$$\int_{0\le y\le x\le 1} \frac{x^2-y^2}{(x^2+y^2)^\alpha} dm. \tag{3}$$

Therefore, the integral (2) exists if and only if the integral (3) is finite. It is the integral of a non-negative function, so the Tonelli Theorem applies: it is equal to the iterated integral

$$\int_0^1 dx \int_0^x \frac{x^2-y^2}{(x^2+y^2)^\alpha} dy.$$

In the interior integral (with respect to y) we make a substitution $y = tx$ to get

$$\int_0^1 x^{3-2\alpha} dx \int_0^1 \frac{1-t^2}{(1+t^2)^\alpha} dt.$$

This integral converges when $3-2\alpha > -1$ or $\alpha < 2$.

Problem 5

Let $0 < \alpha \le 1$. A function $f(x)$ is called Hölder continuous of order α if there exists a constant C such that

$$|f(x) - f(y)| \le C|x-y|^\alpha$$

for all values x and y. Let $f(x)$ be a Hölder continuous function of order α defined on $[0,1]$, and let

$$\Gamma_f = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, y = f(x)\}$$

be the graph of $f(x)$. Prove that the Hausdorff dimension of Γ_f does not exceed $2-\alpha$.

Solution

Let $f(x)$ be a Hölder continuous function of order α. We will show that

$$H_{2-\alpha}(\Gamma_f) < \infty.$$

This fact implies that the Hausdorff dimension of Γ_f is at most $2-\alpha$. Fix $\epsilon > 0$. Let $n > 1/\epsilon$ be an integer number. Let $I_j = [(j-1)/n, j/n]$, $j = 1, \cdots, n$, and let $m_j = \min\{f(x) : x \in I_j\}$, $M_j = \max\{f(x) : x \in I_j\}$. Then

$$\Gamma_f \subset \cup_{j=}^n R_j \quad \text{where} \quad R_j = I_j \times [m_j, M_j].$$

The height of the rectangle R_j equals $M_j - m_j$, and it does not exceed $Cn^{-\alpha}$ because of the Hölder condition. Break each rectangle R_j into $Cn^{1-\alpha}$ rectangles of length $1/n$ and equal height. Their heights are not bigger than $1/n$. Each of these small rectangles can be put inside a circle of radius $1/n < \epsilon$. The total number of these circles is $Cn^{2-\alpha}$. Therefore

$$H_{2-\alpha}(\Gamma_f, \epsilon) \le \left(\frac{1}{n}\right)^{2-\alpha} \times Cn^{2-\alpha} = C,$$

and

$$H_{2-\alpha}(\Gamma_f) = \lim_{\epsilon\to 0} H_{2-\alpha}(\Gamma_f, \epsilon) \le C < \infty.$$