FINAL EXAM FOR MATH 527B
SOLUTIONS

Problem 1

Let \((X, \mathcal{B})\) be a measurable space, and let \(\{f_n(x)\}\) be a sequence of measurable functions on \(X\). Prove that the set of all \(x\) for which \(\lim_{n \to \infty} f_n(x)\) exists is a measurable set.

Solution

Let \(S\) be the set of all \(x\) for which \(\lim_{n \to \infty} f_n(x)\) exists. A sequence of real numbers converges if and only if it is a Cauchy sequence, so

\[
S = \{x : (\forall \epsilon > 0)(\exists N)(\forall n > N)(\forall m > N) |f_n(x) - f_m(x)| < \epsilon\}.
\]

Let

\[
A_{k,m,n} = \left\{x : |f_n(x) - f_m(x)| < \frac{1}{k}\right\}.
\]

The sets \(A_{k,m,n}\) are measurable, so the set

\[
S = \bigcap_{k=1}^\infty \bigcup_{N=1}^\infty \bigcap_{n=N+1}^\infty \bigcap_{m=N+1}^\infty A_{k,m,n}
\]

is also measurable.

Problem 2

Let \((X, \mathcal{B}, \mu)\) be a measure space. One says that a sequence of measurable functions \(\{f_n(x)\}\) converges to a function \(f(x)\) in measure if

\[
\lim_{n \to \infty} \mu\{x : |f_n(x) - f(x)| > \epsilon\} = 0
\]

for every \(\epsilon > 0\). Assume that \(\mu(X) < \infty\). Prove that a sequence \(\{f_n(x)\}\) converges to \(f(x)\) in measure if and only if

\[
\lim_{n \to \infty} \int_X \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \, d\mu = 0.
\]
Solution

Let $S_{n, \epsilon} = \{ x : |f_n(x) - f(x)| > \epsilon \}$. Notice that

$$
\int_X \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \, d\mu \leq \mu(S_{n, \epsilon}) + \epsilon \mu(X).
$$

Here we used the fact that

$$
\frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \leq \min\{1, |f_n(x) - f(x)|\}.
$$

Suppose that $f_n$ converges to $f$ in measure. Given $\delta > 0$, take $\epsilon = \delta/(2\mu(X))$. Then $\mu(S_{n, \epsilon}) < \delta/2$ for sufficiently large $n$, and

$$
\int_X \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \, d\mu < \delta.
$$

This proves that convergence in measure implies (1). To prove that (1) implies convergence in measure we notice that

$$
\mu(S_{n, \epsilon}) \leq \frac{1 + \epsilon}{\epsilon} \int_X \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \, d\mu.
$$

Problem 3

Find the limit

$$
\lim_{n \to \infty} \int_0^\infty \frac{n \sin(x/n)}{x(1 + x^2)} \, dx.
$$

Justify all steps.

Solution

One has

$$
\lim_{n \to \infty} \frac{n \sin(x/n)}{x(1 + x^2)} = \frac{1}{1 + x^2}
$$

for every $x > 0$, and

$$
\left| \frac{n \sin(x/n)}{x(1 + x^2)} \right| \leq \frac{1}{1 + x^2},
$$

so

$$
\lim_{n \to \infty} \int_0^\infty \frac{n \sin(x/n)}{x(1 + x^2)} \, dx = \int_0^\infty \frac{1}{1 + x^2} = \pi/2
$$

by the Dominated Convergence Theorem.

Problem 4

Find all values of $\alpha > 0$ for which the Lebesgue integral

$$
(2) \quad \int_{[0,1] \times [0,1]} \frac{x^2 - y^2}{(x^2 + y^2)^\alpha} \, dm
$$

exists? Here $m$ is the Lebesgue measure. Justify your answer.
Solution

Notice that the integrals of the positive and the negative part of the integrand in (2) are equal to

\[ \int_{0 \leq y \leq x \leq 1} \frac{x^2 - y^2}{(x^2 + y^2)^\alpha} dm. \]

Therefore, the integral (2) exists if and only if the integral (3) is finite. It is the integral of a non-negative function, so the Tonelli Theorem applies: it is equal to the iterated integral

\[ \int_0^1 dx \int_0^x \frac{x^2 - y^2}{(x^2 + y^2)^\alpha} dy. \]

In the interior integral (with respect to \( y \)) we make a substitution \( y = tx \) to get

\[ \int_0^1 x^{3-2\alpha} dx \int_0^1 \frac{1-t^2}{(1 + t^2)^\alpha} dt. \]

This integral converges when \( 3 - 2\alpha > -1 \) or \( \alpha < 2 \).

**Problem 5**

Let \( 0 < \alpha \leq 1 \). A function \( f(x) \) is called Hölder continuous of order \( \alpha \) if there exists a constant \( C \) such that

\[ |f(x) - f(y)| \leq C|x - y|^{\alpha} \]

for all values \( x \) and \( y \). Let \( f(x) \) be a Hölder continuous function of order \( \alpha \) defined on \([0,1]\), and let

\[ \Gamma_f = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, y = f(x)\} \]

be the graph of \( f(x) \). Prove that the Hausdorff dimension of \( \Gamma_f \) does not exceed \( 2 - \alpha \).

**Solution**

Let \( f(x) \) be a Hölder continuous function of order \( \alpha \). We will show that

\[ H_{2-\alpha}(\Gamma_f) < \infty. \]

This fact implies that the Hausdorff dimension of \( \Gamma_f \) is at most \( 2 - \alpha \). Fix \( \epsilon > 0 \). Let \( n > 1/\epsilon \) be an integer number. Let \( I_j = [(j-1)/n, j/n], j = 1, \ldots, n \), and let \( m_j = \min\{f(x) : x \in I_j\}, M_j = \max\{f(x) : x \in I_j\} \). Then

\[ \Gamma_f \subset \bigcup_{j=1}^n R_j \quad \text{where} \quad R_j = I_j \times [m_j, M_j]. \]

The height of the rectangle \( R_j \) equals \( M_j - m_j \), and it does not exceed \( Cn^{-\alpha} \) because of the Hölder condition. Break each rectangle \( R_j \) into \( Cn^{1-\alpha} \) rectangles of length \( 1/n \) and equal height. Their heights are not bigger than \( 1/n \). Each of these small rectangles can be put inside a circle of radius \( 1/n < \epsilon \). The total number of these circles is \( Cn^{2-\alpha} \). Therefore

\[ H_{2-\alpha}(\Gamma_f, \epsilon) \leq \left( \frac{1}{n} \right)^{2-\alpha} \times Cn^{2-\alpha} = C, \]

and

\[ H_{2-\alpha}(\Gamma_f) = \lim_{\epsilon \to 0} H_{2-\alpha}(\Gamma_f, \epsilon) \leq C < \infty. \]