PROBLEM SET 10

Problem 1

Prove that every set that is open in weak topology of l^2 is also open in the norm topology.

Problem 2

Prove that every non-empty set in l^2 that is open in the weak topology is unbounded (a set A is bounded if there exists a constant M such that $||x|| \leq M$ for every $x \in A$.)

Problem 3

Let S be a subset of l^2 that is dense in the norm topology. Let x_n be a bounded sequence in l^2. Prove that if, for some $x \in l^2$, $(x_n, y) \to (x, y)$ for every $y \in S$ then the sequence x_n converges to x in the weak topology. Will the conclusion still be true if one does not assume the sequence x_n to be bounded? Prove of give a counter-example.

Problem 4

Let $f(x)$ be a continuous function on the interval $[-1, 1]$ that satisfies the Holder condition of order $\alpha > 0$ at 0; that means that there exists a constant C such that

$$|f(x) - f(0)| \leq C|X|^{\alpha}.$$

Prove that the following limit

$$\lim_{\epsilon \to 0^+} \left(\int_{-1}^{-\epsilon} \frac{f(x)}{x} dx + \int_{\epsilon}^{1} \frac{f(x)}{x} dx \right) \tag{1}$$

exists. Give an example of a continuous function $f(x)$ (no Holder condition), for which the limit (1) does not esist.