PROBLEM SET 21

Exercises 2.4.23, 2.4.24

Problem 1

Let $f(x)$ be a non-negative function on a measure space $(X, \mathcal{B}, \mu)$. Prove that

$$\mu\{x : f(x) \geq t\} \leq \frac{1}{t} \int_X f(x) d\mu$$

for every $t > 0$.

Problem 2

Let $\Psi(x)$ be a convex (concave up) function.
a) Prove that

$$\Psi(t_1 c_1 + \cdots t_n c_n) \leq t_1 \Psi(c_1) + \cdots t_n \Psi(c_n)$$

for all real numbers t_j, c_j such that $t_j \geq 0$ and $t_1 + \cdots t_n = 1$.
b) Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $\mu(X) = 1$. Prove that

$$\Psi\left(\int_X \phi(x) d\mu \right) \leq \int_X \Psi(\phi(x)) d\mu$$

for every simple function $\phi(x)$ on X.
c) Under the assumptions of part b) prove that

$$\Psi\left(\int_X f(x) d\mu \right) \leq \int_X \Psi(f(x)) d\mu$$

for every non-negative measurable function on X.
d) Let $f(x)$ be a non-negative function on the real line. Prove that

$$\int_{-\infty}^{\infty} f(x) e^{-x^2} dx \leq \sqrt{\pi} \ln\left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{f(x) - x^2} dx \right).$$

Problem 3

Consider the integral

$$\int_1^{\infty} \frac{sinx}{x^{\alpha}} dx.$$

For what values of α it exists as a Lesbegue integral, and for what values of α it converges as an improper Riemann integral?

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