PROBLEM SET 6

Problem 1

Let $C^\infty(\mathbb{R})$ be the set of all infinitely many times differentiable functions on the real line. For a functions $f(x) \in C^\infty(\mathbb{R})$, a non-negative, integer number m, and a number $R \geq 0$ we define

$$N_{m,R}(f) = \max_{0\leq k\leq m} \sup_{|x|\leq R} |f^{(k)}(x)|.$$

Define the distance between two functions from $C^\infty(\mathbb{R})$:

$$d(f,g) = \sum_{m=1}^{\infty} 2^{-m} \frac{N_{m,m}(f-g)}{1+N_{m,m}(f-g)}. \tag{1}$$

a) Show that $C^\infty(\mathbb{R})$, together with the distance (1), form a metric space.
b) Show that sets (3.1.20) on p. 122 form a local base for its metric topology. (Do problem 3.1.21 first).
c) Explain in "simple" terms what the convergence $f_n(x) \to f(x)$ in this topology means (you are not allowed to use the language of metric and topological spaces).

Problem 2

Let $C_0(\mathbb{R})$ be the space of continuous functions $f(x)$ on the real line for which there exists $R > 0$ such that $f(x) = 0$ for $|x| \geq R$ (the number R depends on the function.) Such functions are called functions with compact support. Let $\gamma(x)$ be a continuous, everywhere positive function on $\mathbb{R}$. For $f \in C_0(\mathbb{R})$, we define

$$U(f,\gamma) = \{g \in C_0(\mathbb{R}) : |g(x) - f(x)| < \gamma(x)\}.$$

a) Prove that the collection of the sets $U(f,\gamma)$ over all positive, continuous functions γ form a local base of a topology.
b) Show that a sequence $f_n(x)$ converges to $f(x)$ in this topology if and only if $f_n(x)$ uniformly converges to $f(x)$ and there exists $R > 0$ such that $f_n(x) = 0$ for all n and $|x| \geq R$ (R does not depend on n!).

Problem 3

Let (M,d) be a metric space, and let $x_{j,k}$ be a double sequence, $j,k \geq 1$. Suppose that. for some $x \in M$,

$$\lim_{k\to\infty} x_{j,k} = x$$

for every j. Prove that there exists a sequence of natural numbers n_j such that

$$\lim_{j\to\infty} x_{j,n_j} = x.$$

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Problem 4

Prove that there is no metric on the set $C_0(\mathbb{R})$ that induces the topology from problem 2. *Hint.* You may find the result of problem 3 useful.

Exercises 3.1.21, 3.1.22, 3.1.23