Problem

Find the Hausdorff dimension of the set $S \subset [0,1]$ of numbers the decimal expansion of which does not contain digit 3.

Solution

Let S_n be the set of numbers from $[0,1]$, the first n digits of which are different from 3. The set S_1 is the union of $(0,0.3)$ and $(0.4,1)$. To get S_2, one has to remove from $(0,0.3)$ 3 intervals: $[0.03,0.04]$, $[0.13,0.14]$, and $[0.23,0.24]$; from $(0.4,1)$ one has to remove 6 intervals. The interval $(0,0.3)$ will produce one interval of length 0.03, one interval of length 0.06, and 2 interval of length 0.09. The interval $(0.4,1)$ will produce one interval of length 0.03, one interval of length 0.06, and 5 interval of length 0.09. So, S_2 consists of 2 intervals of length 0.03, 2 intervals of length 0.06, and 7 intervals of length 0.09. Let us show inductively that the set S_n consists of a_n intervals of length 3×10^{-n}, a_n intervals of length 6×10^{-n}, and b_n intervals of length 9×10^{-n}. As one passes from S_n to S_{n+1}, one removes from each interval of length $c10^{-n}$ $(c=3,6.9)$ c intervals of length $!0^{-(n+1)}$; it produces one interval of length $3\times 10^{-(n+1)}$, one interval of length $6\times 10^{-(n+1)}$, and $c-1$ intervals of length $9\times 10^{-(n+1)}$. Therefore S_{n+1} consists of intervals of length $3\times 10^{-(n+1)}$, of length $6\times 10^{-(n+1)}$, and of length $3\times 10^{-(n+1)}$. Moreover, we get recursive relations

$$a_{n+1} = 2a_n + b_n, \quad b_{n+1} = 7a_n + 8b_n.$$

These relations can be written in the form $x_{n+1} = Ax_n$ where

$$x_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 2 & 1 \\ 7 & 8 \end{pmatrix}.$$

The matrix A has eigenvalues 1 and 9; the corresponding eigenvectors are $(1,-1)^t$ and $(1,7)^t$. Therefore,

$$A = D\begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}D^{-1},$$

where

$$D = \begin{pmatrix} 1 & 1 \\ -1 & 7 \end{pmatrix}, \quad \text{and} \quad D^{-1} = \frac{1}{8}\begin{pmatrix} 7 & -1 \\ 1 & 1 \end{pmatrix}.$$

Therefore, $x_n = A^{n-1}x_1 = D\mathrm{diag}(1,9^{n-1})D^{-1}x_1$. One has $x_1 = (0,1)^t$, so

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 1 & 1 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 9^{n-1} \end{pmatrix}\begin{pmatrix} 7 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

this yields

$$a_n = \frac{7+9^{n-1}}{8}, \quad b_n = \frac{7(9^{n-1}-1)}{8}.$$

Let us recall that S_n is the union of a_n intervals of radius 1.5×10^{-n}, a_n intervals of radius 3×10^{-n}, and b_n intervals of radius 4.5×10^{-n}, so

$$\sum r_j^\alpha = a_n(1.5^\alpha + 3^\alpha)10^{-n\alpha} + b_n 4.5^\alpha 10^{-n\alpha}. \tag{1}$$

The quantity on the right in (1) has a finite, non-zero limit when $n\to\infty$ when $10^\alpha = 9$ or

$$\alpha = \log_{10} 9.$$

This is the Hausdorff dimension of S.

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