

to neutron capture in which a compound-nuclear state decays via gamma emission to many low-lying states.¹¹ Agreement of nuclear spectra with GOE theory implies that the capturing state is very complex and, as a result, the total capture width Γ_γ is assumed to be a statistically independent sum of partial widths Γ_{γ_i} to low-lying states, $\Gamma_\gamma = \sum_i \Gamma_{\gamma_i}$. Application of the central limit theorem of statistics to the sum leads to a Gaussian distribution for Γ_γ which is relatively narrow. Similar considerations should apply to the bound states of atoms, especially as the excitation energy increases.

¹C. E. Porter, *Statistical Theories of Spectra: Fluctuations* (Academic, New York, 1965). M. L. Mehta, *Random Matrices and the Statistical Theory of Energy Levels* (Academic, New York, 1967).

²T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, *Rev. Mod. Phys.* **53**, 385 (1981).

³R. U. Hag, A. Pandey, and O. Bohigas, *Phys. Rev. Lett.* **48**, 1086 (1982).

⁴H. I. Liou, H. S. Camarada, S. Wynchank, M. Slagowitz, G. Hacken, F. Rahn, and J. Rainwater, *Phys. Rev. C* **5**, 974 (1972).

⁵G. Hacken, R. Werbin, and J. Rainwater, *Phys. Rev. C* **17**, 43 (1978). This paper contains a complete set of references to earlier papers by the Columbia group.

⁶N. Rosenzweig and C. E. Porter, *Phys. Rev.* **120**, 1698 (1960).

⁷W. C. Martin, Romuald Zalubas, and Lucy Hagan, *Atomic Energy Levels—The Rare-Earth Elements*, U. S. National Bureau of Standards, National Standard Reference Data Series—60 (U.S. GPO, Washington, D.C., 1978).

⁸A $\text{Cov}(S_i, S_{i+1})$ of -0.253 was originally calculated for a 3×3 matrix by C. E. Porter, *Nucl. Phys.* **40**, 167 (1963). For matrices of higher dimension the value tends toward -0.27 (see Ref. 4).

⁹F. Dyson and M. L. Mehta, *J. Math. Phys. (N.Y.)* **4**, 701 (1963).

¹⁰H. S. Camarda, H. I. Liou, F. Rahn, G. Hacken, M. Slagowitz, W. W. Havens, Jr., J. Rainwater, and S. Wynchank, in *Statistical Properties of Nuclei*, edited by J. B. Garg (Plenum, New York, 1972), p. 205.

¹¹L. M. Bollinger, in *Experimental Neutron Resonance Spectroscopy*, edited by J. A. Harvey (Academic, New York, 1970), p. 235.

Propagation of Ultrashort Optical Pulses in Degenerate Laser Amplifiers

I. R. Gabbitov

L. D. Landau Institute for Theoretical Physics, Moscow V334, U. S. S. R.

and

S. V. Manakov

L. D. Landau Institute for Theoretical Physics, Moscow V334, U. S. S. R., and Department of Mathematics, University of Manchester Institute of Science and Technology, Manchester M601QD, United Kingdom

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The shape and area of the output from physically realizable amplifiers of inverted $Q(2)$ -degenerate two-level atoms are calculated analytically. The $(4\pi - \delta)$ - or δ -pulse output is made up from a sequence of $\pm 4\pi$ double-humped pulses followed by $\pm (4\pi - 2\delta)$ pulses, where $\delta = 2 \cos^{-1}(-\frac{1}{3})$.

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The output from very long systems of initially inverted nondegenerate two-level atoms has already been described analytically^{1,2} and was shown to be a π pulse made up of alternating $\pm 2\pi$ pulses. The calculations made use of the complete integrability of the Maxwell-Bloch (MB) envelope and phase equations which apply to both nondegenerate attenuators and amplifiers with suitable changes of sign. This integrability meant that they could be solved by the inverse

scattering method³ in both cases. Results for the attenuator are well known,⁴ but the analysis^{1,2} was the first consistent application of the method to amplifiers in the sharp-line limit.^{5,6} In the degenerate cases, the MB equations are not integrable⁷ and so the inverse method cannot be used. We show in this Letter, nevertheless, how the output from an inverted two-level atomic medium can be calculated whether or not the atomic transition is degenerate.

The problem can still be solved in both cases because the similarity solution is the important part. In the nondegenerate case^{1,2} the inverse method showed that the similarity solution was involved in a way independent of the shape of the input pulse together with a part depending only on a sufficiently small region at the front of the input pulse and therefore depending linearly upon it. The input influenced the output only in this way and we called the resultant solution "semi-self-similar." Here we show that semi-self-similar solutions are obtained for both degenerate and nondegenerate amplifiers and that the inverse method need not be used to find them. In any case, the inverse method most easily finds solutions for amplifiers of physically impractical lengths l such that $\ln(l/l_0) \gg 1$ while l_0 proves (for solid-state amplifiers for example, see below) to be about 10 cm. The methods used here readily apply to amplifiers in the physically accessible regime where $l \simeq l_0$.

Similarity solutions of the MB equations have been investigated for π pulses in nondegenerate amplifiers^{5,6} and in the theory of superfluorescence.⁸ Degeneracy introduces qualitatively new features: In the case of $Q(2)$ symmetry there are four attenuators and four amplifiers⁹ and 4π , 2δ , $(4\pi - 2\delta)$, and 0π pulses can propagate depending on initial conditions in agreement with the area theorem^{4,5,7,9}, $\delta = 2 \cos^{-1}(-\frac{1}{4})$. Superfluorescent pulses from totally inverted $Q(2)$ -degenerate systems are δ or $(4\pi - \delta)$ pulses not π pulses.¹⁰ In this Letter we study sharp-line, strictly resonant, optical pulse propagation in amplifiers containing $Q(2)$ -degenerate two-level atoms. We study both the physical regime $l \simeq l_0$ and, for mathematical reasons, the regime where $\ln(l/l_0) \gg 1$. We find that oscillatory δ and $(4\pi - \delta)$ pulses propagate for $l \simeq l_0$. But for $\ln(l/l_0) \gg 1$ these can be seen

to consist of trains of alternating $\pm 4\pi$ pulses followed by $\pm(4\pi - 2\delta)$ pulses, and these features are retained to some degree as $l \rightarrow l_0$.

For $Q(2)$ symmetry there are two Bloch equations coupled through the incident field envelope $E(x, t)$.^{5,7,9,10} At strict resonance in the sharp-line limit the MB equations reduce to^{5,7,11} (with $\varphi_{tt} \equiv \partial^2 \varphi / \partial t^2$, etc.)

$$\varphi_{tt} + \varphi_{tx} = 4\Omega_0^2 \gamma^{-1} (a \sin \varphi + \frac{1}{2} b \sin \frac{1}{2} \varphi), \quad (1)$$

with $E(x, t) = (1/2p)\varphi_t$. We use $\Omega_0^2 \equiv \frac{1}{2}\pi\omega_s p^2 \gamma$; $\gamma \equiv a + \frac{1}{4}b$, $a = \sigma_{-2} + \sigma_2$, $b = \sigma_{-1} + \sigma_1$, $\sigma_m = N_m^+ - N_m^-$; ω_s is the frequency of the transition; the magnitudes of the nonvanishing matrix elements are $\frac{1}{2}|m|p$ ($m = \pm 2, \pm 1$); and N_m^+ are the initial numbers of atoms per unit volume in their upper states labeled by $m = \pm 1$ or ± 2 ; N_m^- are the numbers in their lower states.¹² The mathematical problem to be solved is as follows: For $t < 0$, $E \equiv 0$ in the half-space $x > 0$ occupied by atoms. The field E enters at $x = 0$ at $t = 0$. Given $E(0, t) = 0$, $t < 0$, and $E(0, t) = E_0(t)$, $t \geq 0$, we want to find $E(x, t)$ for $x > 0$. We suppose the front of the incoming pulses is such that $E_0(t) = E_\nu(t\tau^{-1})^\nu [1 + O(t\tau^{-1})]$; τ and ν are real and positive and τ is a rise time. For large enough x this is all the information that we need about the input at $x = 0$ in order to determine the output at x . From $E(x, t) = (1/2p)\varphi_t$ the corresponding information on φ is

$$\varphi(0, t) = \varphi_0(t) \equiv 2p\tau(\nu + 1)^{-1} E_\nu(t\tau^{-1})^{\nu+1}, \quad (2)$$

and the boundary and initial-value problem for (1) is $\varphi(0, t) = \varphi_0(t)$; $\varphi(x, 0) = 0$, $x > 0$. This is just the Cauchy problem $\varphi(x, 0) = 0$, $x > 0$; $\varphi(0, t) = \varphi_0(t)$, $t > 0$ in which t and x interchange.

Since φ_0 describes the front of the input pulse, it is small and $\sin \varphi_0 \simeq \varphi_0$. Equation (1) can now be solved near the light cone $x = t$ in linear approximation, and

$$\varphi(x, t) = 8\Omega_0^2 x z^{-1} \int_{\mathcal{C}} \tilde{\varphi}_0(8\Omega_0^2 x z^{-1} \lambda) \exp\{-i\frac{1}{2}(\lambda - \lambda^{-1})z\} d\lambda, \quad (3)$$

where $z = 4\Omega_0[x(t-x)]^{1/2}$, $t > x$, and $\tilde{\varphi}_0(\omega)$ is the Fourier transform of $\varphi_0(t)$. The path of integration \mathcal{C} joins $\lambda = -\infty$ to $\lambda = +\infty$ through the upper half-plane so that (3) is causal; and because $\varphi_0(t) = 0$, $t < 0$, $\tilde{\varphi}_0(\lambda)$ is analytic in the upper half-plane while $\tilde{\varphi}_0 \sim \lambda^{-(\nu+2)}$ for large λ . Thus (3) can be evaluated by stationary phase as

$$\begin{aligned} \varphi(x, t) &= 8\Omega_0^2 x z^{-1} \tilde{\varphi}_0(8\Omega_0^2 x i z^{-1}) \{e^z (2\pi z)^{-1/2}\} [1 + O(z^{-1})] \\ &= E_\nu \Gamma(\nu + 1) 2p\tau (8\Omega_0^2 \tau x z^{-1})^{-\nu-1} \{e^z (2\pi z)^{-1/2}\} [1 + O(z^{-1})], \end{aligned} \quad (4)$$

and this is valid for $\varphi(x, t) \ll 1$ and so for large enough x . The similarity variable $z = 4\Omega_0[x(t-x)]^{1/2}$ appears alone in the curly brackets, while the input data (2) influence the solution only through the factor $(zx^{-1})^{\nu+1}$. This combination of factors is characteristic of the semi-self-similar solutions.^{1,2} When

the input is $E_0(t) = E_{-1} \delta(t \tau^{-1})$, $\nu = -1$, and (4) depends *only* on the self-similar solution.

If we set $\xi \equiv \ln(zx^{-1}\Omega_0^{-1})$ the basic equation (1) becomes

$$\varphi_{zz} + z^{-1}\varphi_z - z^{-2}\varphi_{\xi\xi} = \gamma^{-1}(a \sin \varphi + \frac{1}{2}b \sin \frac{1}{2}\varphi) \quad (5)$$

and solutions independent of ξ are self-similar, satisfying

$$\varphi_{zz} + z^{-1}\varphi_z = \gamma^{-1}(a \sin \varphi + \frac{1}{2}b \sin \frac{1}{2}\varphi). \quad (6)$$

Self-similar solutions regular at $z=0$ form a one-parameter family $\psi = \psi(\psi_0, z)$ determined by the value of ψ at $z=0$: $\psi(\psi_0, 0) = \psi_0$. When $\psi_0 \ll 1$, $\psi(\psi_0, z) = \psi_0 I_0(z)$ and $I_0(z)$ is the Bessel function of imaginary argument.^{5,6,8} For large z , $I_0(z) \sim e^z \times (2\pi z)^{-1/2} [1 + O(z^{-1})]$ and therefore ψ is $O(1)$ only for $z \sim \ln |\psi_0|^{-1}$ and the self-similar solution differs significantly from zero only for $z \gg 1$, so that $z^{-2}\varphi_{\xi\xi}$ in (5) can be dropped and (5) reduces to (6). This means that the solution of (5) is $\varphi(x, t) = \psi[\psi_0(zx^{-1}), z]$ and ψ_0 is determined by the compatibility of (4) with the asymptotic form $\psi(\psi_0, z) \sim \psi_0 e^z (2\pi z)^{-1/2}$ of the linearized self-similar solutions. Plainly,

$$\psi_0(zx^{-1}) = E_\nu \Gamma(\nu + 1) 2p\tau (z/8\Omega_0^2 \tau x)^\nu \quad (7)$$

which completes the solution. It is valid for

$$1 \ll z < \ln[(8\Omega_0^2 x \tau)^\nu / 2\Gamma(\nu + 1)\tau p E_\nu] \equiv \ln d_\nu,$$

a condition we discuss below. Thus it remains only to find the self-similar solutions $\psi(\psi_0, z)$ which are solutions of the ordinary differential equation (6).

The nondegenerate case of (6) is $b=0$, $a = +n$, the number of atoms per unit volume, and the results in the physical regime are well known.^{5,6,8} The pulse is a π pulse made up of successive damped oscillations^{5,6,13}: The distance between successive peaks $\sim \ln \ln d_\nu$. Thus the oscillations are resolved but substantially damped for realistic amplifiers for which $1 \ll \ln \ln d_\nu$, while in the nonphysical regime for which $1 \ll \ln \ln d_\nu$ the output becomes a series of isolated pulses initially each of area $\pm 2\pi$ with speed > 1 ($> c$).^{1,2,14} The net area of this sequence remains π and a front which "piles up" develops in the usual way.¹³ Some of these features should be observable in the physical regime and in particular the areas of the leading pulses will be close to 2π .^{15,16}

A new feature is present in the degenerate case. Equation (6) formally describes the motion of a particle in the potential $U(\varphi) = a \cos \varphi + b \cos \frac{1}{2}\varphi$ with $4a > b > 0$ and $0 \leq \varphi \leq 4\pi$: The $z^{-1}\varphi_z$ is then

a "damping" term. There are degenerate minima at $\delta = 2 \cos^{-1}(-\frac{1}{4}ba^{-1})$ and $4\pi - \delta$, and the damping means that φ , which starts at $\varphi = \psi_0 \ll 1$, eventually reaches δ or $4\pi - \delta$ and the pulse areas are δ or $4\pi - \delta$. Numerical results for φ in the realistic case $a = b$ are given in Fig. 1. The fields are calculated from φ_z (shown in inset). The form of $U(\varphi)$ means that there are critical values such that if $\psi_0^{(2k)} < \psi_0 < \psi_0^{(2k+1)}$ the system reaches $\varphi = \delta$ and otherwise $4\pi - \delta$. Figure 1 shows that if $\psi_0 = 10^{-4}, 10^{-8}$ then ψ reaches δ ; if $\psi_0 = 10^{-6}, 10^{-10}$ ψ reaches $4\pi - \delta$. But the solution (6) depends on $\psi_0(zx^{-1})$. So that, as z (i.e., t) changes at fixed x , $\varphi(x, t)$ itself jumps by $\pm(4\pi - 2\delta)$. Near such $\psi_0^{(k)}$ the derivatives $\varphi_{\xi\xi}$ are large and $z^{-2}\varphi_{\xi\xi}$ in (5) is not negligible. Still, jumps plainly occur at some definite ξ_k 's and $\psi_0(\xi_k) = \psi_0^{(k)}$. Then since ξ depends only on $x t^{-1}$ each such jump has a velocity V_k given by $e^{\xi_k} = 4(V_k^{-1} - 1)^{1/2}$ and $V_k < 1$.

Figure 1 shows that, for small enough ψ_0 , at least one double-humped pulse determined by $\psi_z(\psi_0, z)$ of area $\approx 4\pi$ will be emitted. This corresponds to the exact but unstable 4π -pulse solution for the $Q(2)$ -degenerate "full amplifier" whose speed is > 1 ,⁹ but its area is $< 4\pi$ and it is asymmetric through the damping $z^{-1}\varphi_z$ in (6). Thus in the nonphysical regime $\ln \ln d_\nu \gg 1$ the output is a series of $\pm 4\pi$ -type asymmetric double-humped pulses traveling at speeds > 1 and piling up. The

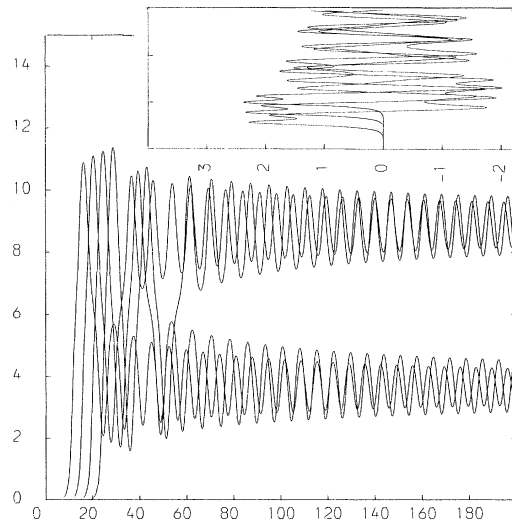


FIG. 1. Similarly solutions $\psi(\psi_0, z)$, solutions of Eq. (6), against z for $\psi_0 = 10^{-4} - 10^{-10}$. Note the two equilibrium states $\delta = 2 \cos^{-1}(-\frac{1}{4})$ and $4\pi - \delta$; for $\psi_0 = 10^{-4}, 10^{-8}$ ψ converges on δ ; for $\psi_0 = 10^{-6}, 10^{-10}$ ψ converges on $4\pi - \delta$. Inset is $\psi_z(\psi_0, z)$ for $\psi_0 = 10^{-4}, 10^{-6}, 10^{-8}$.

sequence damps out and the system settles on to a $(4\pi - \delta)$ or δ attenuator as defined in Ref. 9. These attenuators propagate $\pm(4\pi - 2\delta)$ pulses with speeds $V_k < 1$ and the new feature of the $Q(2)$ -degenerate amplifier is that a sequence of these pulses now propagates, apparently without damping, but with pulse spacing increasing as $\ln \ln d_v$.¹⁴

In the physically realizable regime for which $\ln d_v \gg 1$ (Ref. 14) this sequence of pulses for $E(x, t)$ becomes a damped oscillatory wave train headed by one or more discernibly double-humped asymmetric pulses of the 4π type characteristic of $Q(2)$ symmetry and this will still be followed by jumps of $\pm(4\pi - 2\delta)$ which can be seen in a plot of $\varphi(x, t) = 2p \int_{-\infty}^t E(x, t') dt'$. For $Q(2)$ -degenerate iodine on the $2P_{1/2} \rightarrow 2P_{3/2}$ ($F=2 \rightarrow F=2$) transition, $\ln d_v = 1$ (and no better) for $x = 100$ cm and Rabi frequency $pE \sim 10^9$.¹⁴ But for solid-state amplifiers, $\ln d_v \approx 10$ for $1 \lesssim x \lesssim 100$ cm.

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¹S. V. Manakov, Pis'ma Zh. Eksp. Teor. Fiz. **35**, 193 (1982) [JETP Lett. **35**, 237 (1982)].

²S. V. Manakov, Zh. Eksp. Teor. Fiz. **83**, 68 (1982) [Sov. Phys. JETP **83**, 37 (1982)].

³M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Lett. **31**, 125 (1973), and, for example, *Solitons*, Topics in Current Physics, Vol. 17, edited by R. K. Bullough and P. J. Caudrey (Springer-Verlag, Heidelberg, 1980), Chap. I and later chapters.

⁴S. L. McCall and E. L. Hahn, Phys. Rev. Lett. **18**, 908 (1967), and Phys. Rev. **183**, 457 (1969); C. K. Rhodes, A. Szöke, and A. Javan, Phys. Rev. Lett. **21**, 1151 (1968); especially for self-induced transparency and the "area theorem." Soliton theory is in, e.g., *Solitons*, Ref. 3, Chap. II and VI, and in P. J. Caudrey, J. D. Gibbon, and R. K. Bullough, J. Phys. **6**, 1337 (1973); G. L. Lamb, Jr., Phys. Rev. A **9**, 422 (1974); M. J. Ablowitz, D. J. Kaup, and A. C. Newell, J. Math. Phys. **15**, 1852 (1974); R. K. Bullough, in *Interaction of Radiation with Condensed Matter* (IAEA, Vienna, 1977), Vol. I, Report No. IAEA-SMR-20/51, and refer-

ences therein; D. J. Kaup, Phys. Rev. A **16**, 704 (1977), and other references.

⁵G. L. Lamb, Jr., Rev. Mod. Phys. **43**, 99 (1971).

⁶G. L. Lamb, Jr., Phys. Rev. A **12**, 2052 (1975).

⁷R. K. Bullough and P. J. Caudrey, *Solitons*, Ref. 3, Chap. III.

⁸D. C. Burnham and R. Y. Chiao, Phys. Rev. **188**, 667 (1969); R. K. Bullough, P. M. Jack, P. W. Kitchen-side, and R. Saunders, Phys. Scr. **20**, 364 (1979), and references therein. For superfluorescence see, e.g., *Cooperative Effects in Matter and Radiation*, edited by C. M. Bowden, D. W. Howgate, and H. R. Robl (Plenum, New York, 1978).

⁹S. Duckworth, R. K. Bullough, P. J. Caudrey, and J. D. Gibbon, Phys. Lett. **57A**, 19 (1976); Bullough, Ref. 4.

¹⁰J. A. Hermann and R. K. Bullough, J. Opt. Soc. Am. **78**, 701 (1978), and to be published.

¹¹Equation (1) reduces to the problem of $Q(2)$ -degenerate self-induced transparency for n atoms/cm³ treated in Refs. 7 and 9 [the "full attenuator" of the first of Ref. 9] when $\sigma_m = \sigma_{-m}$, $a = b = -n$. It reduces to the "full amplifier" of Ref. 9 when $a = b = +n$.

¹²We assume incoherent pumping and the initial steady state of atoms in upper and ground states reflects this. Pumping is on the time scale of the inhomogeneous broadening, supposed long compared with pulse lifetimes, so that both are neglected.

¹³A. Içsevçi and W. E. Lamb, Jr., Phys. Rev. **185**, 517 (1969).

¹⁴The condition $\ln \ln d_v \gg 1$ is the condition $\ln(l/l_0) \gg 1$ first given and distinguishes the mathematically interesting cases of very well separated pulses from the physically realizable cases $\ln d_v \gg 1$. The number $d_v \lesssim 1$ for a vapor [such as the $2P_{1/2} \rightarrow 2P_{3/2}$ ($F=2 \rightarrow F=2$) transition in iodine]. For $\Omega_0^2 \sim 4 \times 10^{15} \text{ sec}^{-2}$ with $n \sim 10^{16} \text{ cm}^{-3}$, and for a Rabi frequency $pE \hbar^{-1} \sim 10^8$ and $c = 3 \times 10^{10} \text{ cm sec}^{-1}$, $x^{-1} d_v \sim 10^{-2}$ and, for $x \equiv l \sim 10^2$ cm, $d_v \sim 1$ (so $l_0 \sim 10^2$ cm). For solid-state amplifiers with parameters similar to ruby, $n \sim 10^{21} \text{ cm}^{-3}$ or better and $\ln d_v \approx 10$ for $1 \lesssim l_0 \lesssim 100$ cm. The number d_v^{-1} plays the role of a "tipping angle" in the semiclassical theory of superfluorescence (Ref. 8). The amplified pulse emerges with a delay time $\tau_D = \tau_{SF} \ln d_v$, where $\tau_{SF} = 8\Omega_0^2 L c^{-1}$, so that τ_D is the superfluorescence delay time [Ref. 8; J. A. Hermann, Phys. Lett. **69A**, 316 (1979); J. C. MacGillivray and M. S. Feld, in *Cooperative Effects in Matter and Radiation*, edited by C. M. Bowden, D. W. Howgate, and H. R. Robl (Plenum, New York, 1978)]. The pulse separation in time is $\tau_{SF} \times \ln \ln d_v$ (compare Hermann). The close connection between this theory of the amplifier and the theory of superfluorescence will be established elsewhere.

¹⁵Hermann, Ref. 14.

¹⁶See MacGillivray and Feld, Ref. 14.