Modeling “Invisibility” by Optical Cloaking of a Sphere

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Abstract

Modeling invisibility, or optical cloaking has become a fascinating topic in recent years. We used this project as an opportunity to gain knowledge in the method of invisibility cloaking as well as a chance to attempt to write a coordinate transformation in Matlab that would accurately recreate an existing model of a sphere being cloaked. We manipulated Maxwell’s equations under coordinate transformations to do so. Using these methods, we were able to successfully obtain a model in which light rays approach a spherical cloak and redirect around it to reconvene to their original path on the other side, making the sphere invisible. Invisibility is very useful in many fields because it can be used for a variety of applications; the options available with cloaking objects and making them invisible are limitless.
Introduction and Background

Invisibility

The concept of invisibility has always been a very appealing topic. The foundation for invisibility lies in the idea of the redirection of light rays around an object in such a manner that the ray appears on the opposite side of the object. It exits at the same angle as it approaches, as if the ray traveled straight through the object. Thus, the observer only “sees” what is behind the object. Recent advancements in nanotechnology have made it possible for scientists to create materials that are capable of manipulating the behavior of electromagnetic waves. Through experimentation, these materials, called metamaterials, have been formulated for different wavelengths. With research, these materials can be designed with finer structures, to affect wavelengths as small as visible light. It is our goal to formulate a mathematical model that represents a metamaterial that optically cloaks a spherical object.

Metamaterials

By studying the behavior of light rays traveling through different mediums, metamaterials have been developed. There are many notable properties that are associated with metamaterials. They are not found anywhere in nature, thus they must be artificially engineered in a laboratory. They generally exhibit a negative index of refraction. The index of refraction may be represented mathematically by $n$ in the following equation:

$$n^2 = \varepsilon \mu$$  

(1)
where \( \mu \) represents the magnetic permeability and \( \epsilon \) is the electrical permittivity.

We will see these terms again when we introduce Maxwell’s equations. One can see that the index of refraction depends solely on these parameters. Solving equation 1 yields two solutions, one positive and one negative. All materials in nature exhibit a positive index of refraction. Through testing and the development of nanomaterials, negative refractive indices have been created. It is so difficult to design materials to cloak visible light because of its small wavelength. The basic structure of these materials have to be engineered so that they are much smaller than the wave length of the type of wave they are trying to manipulate. For example, radio waves can be successfully cloaked because they have wavelengths that range anywhere from 1 to 10 meters. Designing a structure to cloak visible light, which ranges on the electromagnetic spectrum between 400 and 700 nanometers is much more intricate.

**Mathematical Model**

In order to simulate optical cloaking, we investigated the results of “Taking the wraps of cloaking” by John Pendry [1]. This article suggests manipulating equations that describe properties of electromagnetic behavior and subduing them to coordinate transformations. A Poynting vector can be used to represent the lines of force in a magnetic or electric field. Visualize an object in plane, with all the Poynting vectors stuck to that plane, that undergoes continuous coordinate deformations. The vectors change position in relation to each other, but remain in the same space of the original plane. An illustration of this concept may be found in
the figure below. This is how our electromagnetic properties may remain under coordinate transformations.

Poynting Vector Traveling Through Compressed Medium

**Numerical Methods**

**Maxwell’s Equations**

The basis for constructing metamaterials resides in a thorough understanding of Maxwell’s equations and the behavior of magnetic and electric fields. Michael Faraday presented a good understanding of magnetic behavior by placing iron filaments near a magnet and observing the reaction the fillings had to the magnetic field. The lines that the filaments create physically represent the magnetic field lines. The results of Faraday’s observations are represented in Maxwell’s equations:

\[ \nabla \times E = -\mu(r)\mu_0\frac{\partial H}{\partial t} \quad (2) \]
\[ \nabla \times H = +\varepsilon(r)\varepsilon_0\frac{\partial E}{\partial t} \quad (3) \]

where \( \mu(r) \) represents the magnetic permeability and \( \varepsilon(r) \) is the electrical permittivity. The curl of the electric, \( E \), and magnetizing, \( H \), fields are given by the terms on the left hand side of the equations. Physically, the magnetizing field may be thought of as the magnetic intensity. It is important to note that the magnetizing field differs from the magnetic field. One can see from equation 2 that the curl of the
The electric field is negatively proportional to the first time derivative of the magnetizing field, and also depends on the magnetic permeability. Thus, as the intensity of the magnetic field increases with time, the curl of the electric field decreases. Similarly, the curl of the magnetizing field is proportional to the time derivative of the electric field and the electrical permittivity. This has an opposite effect for the magnetizing field, increasing with an increase in the electric field.

Maxwell’s equations provide to be very applicable for the problem at hand since they do not change under coordinate transformations. Thus, any coordinate system may be used to describe the object, making it possible for all geometries to be considered. In our case, we wish to model the redirection of light rays around a spherical object for simplicity. This way the simulated waves behave the same on any surface of the object. This will require that we transform equations 2 and 3 into spherical coordinates, while retaining our values for \( \mu \) and \( \varepsilon \).

**Coordinate Transformation**

The following derivation of the spherical coordinate transformation was adopted from Pendry [1]. We begin by considering waves traveling through space. These waves can be represented by the following equations, where \( k_0 \) is the free space wave vectors and \( k' \) is the vector in the compressed region:

\[
\begin{align*}
k' md &= k_0 d \quad \text{(4)} \\
k' &= k_0 \sqrt{\varepsilon \mu_y} \quad \text{(5)}
\end{align*}
\]
With $m$ being the compression factor of the medium and $d$ is the thickness of the layer the vector is traveling through. The subscripts on $\mu$ and $\varepsilon$ represent with which axis the tensors are parallel to. Solving for the magnetic permeability and electric permittivity:

$$\varepsilon_y = \mu_y = m^{-1} \quad (6)$$

Similarly for the uncompressed region:

$$k'' = k_0 \sqrt{\varepsilon_y \mu_x} \quad (6)$$

The condition that $k'' = k_0$ yields that:

$$\varepsilon_y = \mu_y = m \quad (7)$$

These values of $\mu$ and $\varepsilon$ are essential for physically interpreting the results of our mathematical model. Pendry's coordinate transformation is as follows:

$$r' = (R_2 - R_1) r / R_2 + R_1, \quad \theta' = \theta, \quad \phi' = \phi \quad (8)$$

$$\varepsilon r' = \mu r' = [R_2 / (R_2 - R_1)] [(r' - R_1) / r']^2, \quad R_1 > r' > R_2 \quad (9)$$

$$\varepsilon \theta' = \mu \phi' = \varepsilon \phi' = \mu \phi' = R_2 / (R_2 - R_1). \quad (10)$$

Where $R_1$ is the inner radius and contains the object and $R_2$ is the outer radius which contains all the variation of the vectors in the transformation. The waves must only distort the coordinate grid where $R_1 < r < R_2$, so this is the only region affected by the different $\mu$ and $\varepsilon$ values described in equations 9 and 10.
We wish for the light ray approaching the object of radius $R_1$ to be redirected about the object, but not to distort the grid outside of the radius $R_2$. This region between $R_1$ and $R_2$ is the only space where the Poynting vectors change position.

**Numerical Results**

**Experimentation**

Using the previously presented coordinate transformation, we aimed to simulate the redirection of rays about a sphere using Matlab. We will define our sphere that is to be cloaked to be centered at the origin in Cartesian coordinates with a radius of 0.5:

$$x^2 + y^2 + z^2 = 0.25$$  \hspace{1cm} (11)

Our code defined our sphere as follows:

$$[x, y, z] = \text{sphere}$$
xs1 = 5*x;
ys1 = 5*y;
zs1 = 5*z;

Now, we want to implement the transformation in equations 8-10 in a function. We want to define our spherical coordinate system, and then manipulate our values for z and r, defining our own parameters for R1 and R2. The first statement converts to spherical coordinates. The if statement of the following code implements equation 8:

\[ r_1(i) = \sqrt{x_1(i)^2 + y_1(i)^2 + z_1(i)^2}; \]
\[ \theta_1(i) = \arccos\left(\frac{z_1(i)}{r_1(i)}\right); \]

\[ \text{if } r_1(i) \leq 1 \]
\[ R_1(i) = 0.5 \times r_1(i) + 0.5; \]
\[ \text{else} \]
\[ R_1(i) = r_1(i); \]
\[ \text{end} \]

We had to be sure to account for small programming errors, such as dividing by zero or limiting our parameters, before moving on. Next, we converted back into Cartesian coordinates for plotting:

\[ X_1(i) = R_1(i) \times \sin(\theta_1(i)) \times \cos(\phi_1(i)); \]
\[ Y_1(i) = R_1(i) \times \sin(\theta_1(i)) \times \sin(\phi_1(i)); \]
\[ Z_1(i) = R_1(i) \times \cos(\theta_1(i)); \]

Next, we wanted to replicate “light rays” by defining three dimensional space where we could create parameterized equations that would finitely plot points,
Creating our lines. The following excerpt is a single ray given in a for loop to illustrate how we chose our points in space:

```matlab
for i = 1 : 4000
    x1(i) = 4 - i / 500;
    y1(i) = 0;
    z1(i) = 0;
```

In this example, our light ray begins at point (4,0,0) in (x,y,z) space. Notice that the ray is only moving along the x direction, and the loop will increment until it reaches the point (-4,0,0). We initially chose to extend 4 units in each direction so that we would be far enough outside of our sphere of radius 0.5 that we could observe the incident angle on each side. Also, the ray will travel through the center of the sphere so we could observe symmetry. By defining multiple arrays throughout the loop, multiple lines could be created. We were able to specifically choose our beginning and ending points so that we could observe vectors approaching from all angles. We were hesitant to create more than one light ray at a time though, as we immediately ran into problems whenever a simulated ray approached our sphere directly normal to its surface. The following images resulted when we ran the code going directly through the middle of the sphere along the x-axis. The concentric circles surrounding the sphere were created to contain our outer radius:
Simulated ray tracing exactly through the center of the sphere

Ray through center of sphere in x-y plane

It can be seen clearly that our first test run did not undergo the transformation. It occurred to us that maybe symmetry would produce unusual results under cylindrical coordinates, so we applied another ray that was off center, going from (4,0.25,0.25) to (-4,0.25,0.25).
Our light ray appeared to undergo the coordinate transformation completely, all while being retained inside of the outer radius. We continued to experiment and eventually we implemented parameterized rays that advanced in every direction with $i$, moving in three-dimensional space at once. The following images express our progression in coding as we experimented with rays at different angles:

No rays are redirected outside concentric circles; all transformation is kept within our defined boundaries.
Observe the ray approaching from the negative z axis at the normal is being absorbed by the sphere.
It does not reappear on the top of the sphere.

It is important to notice the exit angle of each ray compared to that which it entered. To test that they were the same, we plotted the same ray twice at the same time, once before undergoing the transformation. All parameterizations tested lined up both before and after the coordinate manipulation.
Final Results

After tweaking the code to produce symmetrical rays we created our final images that successfully illustrate the coordinate transformation. We made the light rays symmetrical about the x axis in our final illustrations so that our sphere looks nice. It is obvious that they are undergoing the correct coordinate transformation when you observe from different angles.
Conclusion

Although we were unable to correct the problem of our rays approaching at the normal not undergoing the transformation, we found it to be negligible. In our research of other experimentation done with invisibility cloaking, we found that in no material was there any mention of this problem occurring. If we were to simulate infinitely many rays at the object, there would only be a small percentage that would approach exactly normal and go through the object compared to those hitting our outer radius at various angles. So if one’s eye were looking at the object, their eye would be directed towards all of the light rays which were being redirected around it rather than looking at the very small portion of the rays which penetrate it. Also, we had no explanation for the affect when the rays came from the z-direction and hit the sphere at the normal but did not penetrate it and stopped. We have yet to decide what could be causing this but found it also to be negligible because the rays that are coming from that direction in reality would be coming up from the ground where a light source would not be present. Physically, if we were able to create a metamaterial that had the same properties as our coordinate transformation, invisibility would still be achieved. Our results were successful in recreating the results of the Pendry paper.

Future Work

There are many practical applications for optical cloaking. Of course, the military would benefit greatly from a material that was able to hide objects from visible light. By manipulating other wavelengths, such as radio waves, objects could
be invisible to radar detection, which could also prove to be very beneficial. Right now the military is working on cloaking spaces that would be invisible to the radar of planes, which fly overhead and survey particular areas of interest. If the particular area of interest were to be cloaked at the level which made it invisible to radar then such planes would fly over the area not able to detect the location underneath itself. Optical black holes are an interesting topic of study as well. Instead of directing light rays around an object, optical black holes direct all light rays to a single point. If this property were to be applied to light waves, a transformer could convert light energy into power, creating, for example, an extremely efficient solar panel. If this were possible the dimensions of solar panels could be reduced to minute sizes compared to what they are currently at right now. Although metamaterials have not yet publically been designed that alter all spectrums of visible light, the technology is present and rapidly advancing. With the increased study of nanomaterials and electromagnetic interaction with these materials, invisibility is a real possibility in the near future.

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References
