Dynamics of the Elastic Pendulum

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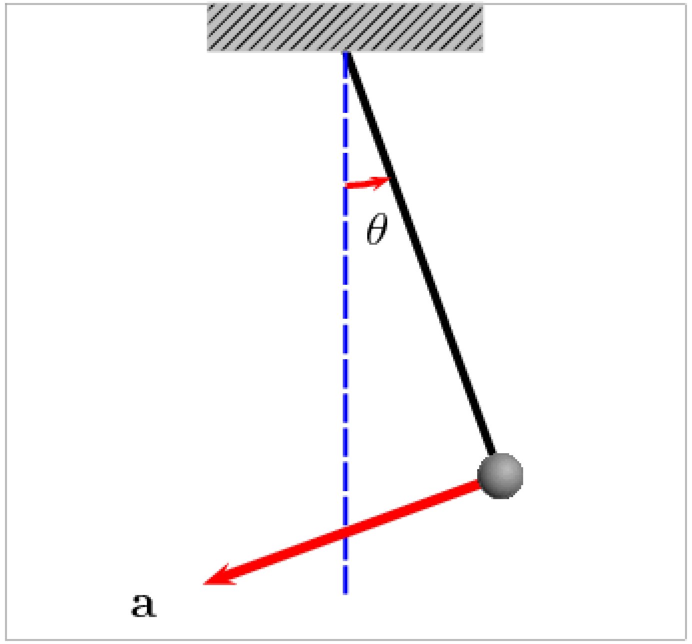
Agenda

- Introduction to the elastic pendulum problem
- Derivations of the equations of motion
- Real-life examples of an elastic pendulum
- Trivial cases & equilibrium states
- MATLAB models

The Elastic Problem (Simple Harmonic Motion)

- $F_{net} = m\frac{d^2x}{dt^2} = -kx \Rightarrow \frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x$
- Solve this differential equation to find
$$x(t) = c_1\cos(\omega t) + c_2\sin(\omega t) = Acos(\omega t - \varphi)$$
- With velocity and acceleration
$$v(t) = -A\omega\sin(\omega t + \varphi)$$
$$a(t) = -A\omega^2\cos(\omega t + \varphi)$$
- Total energy of the system
$$E = K(t) + U(t)$$
$$= \frac{1}{2}mvt^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

The Pendulum Problem (with some assumptions)



- With position vector of point mass $\vec{x} = l\,(sin\theta\vec{\imath} - cos\theta\vec{\jmath})$, define $\vec{r}$ such that $\vec{x} = l\vec{r}$ and $\vec{\theta} = cos\theta\vec{\imath} + sin\theta\vec{\jmath}$
- Find the first and second derivatives of the position vector:

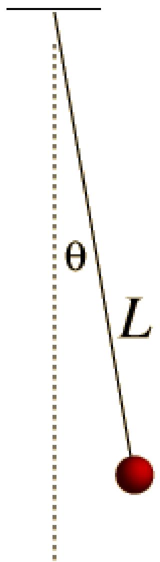
$$\frac{d\vec{x}}{dt} = l\frac{d\theta}{dt}\vec{\theta}$$

$$\frac{d^2\vec{x}}{dt^2} = l\frac{d^2\theta}{dt^2}\vec{\theta} - l\left(\frac{d\theta}{dt}\right)^2\vec{r}$$

- From Newton's Law, (neglecting frictional force)

$$m\frac{d^2\vec{x}}{dt^2} = \overrightarrow{F_g} + \overrightarrow{F_t}$$

The Pendulum Problem (with some assumptions)



Defining force of gravity as $\overrightarrow{F_g} = -mg\vec{j} = mgcos\theta\vec{r} - mgsin\theta\vec{\theta}$ and tension of the string as $\overrightarrow{F_t} = -T\vec{r}$:

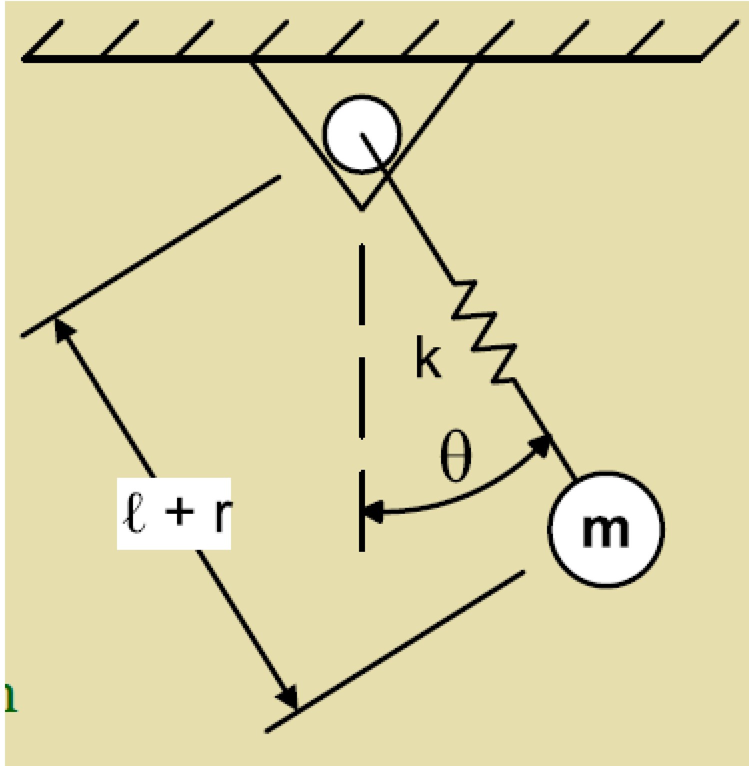
$$-ml\left(\frac{d\theta}{dt}\right)^2 = mgcos\theta - T$$

$$ml\frac{d^2\theta}{dt^2} = -mgsin\theta$$

Define $\omega_0 = \sqrt{g/l}$ to find the solution:

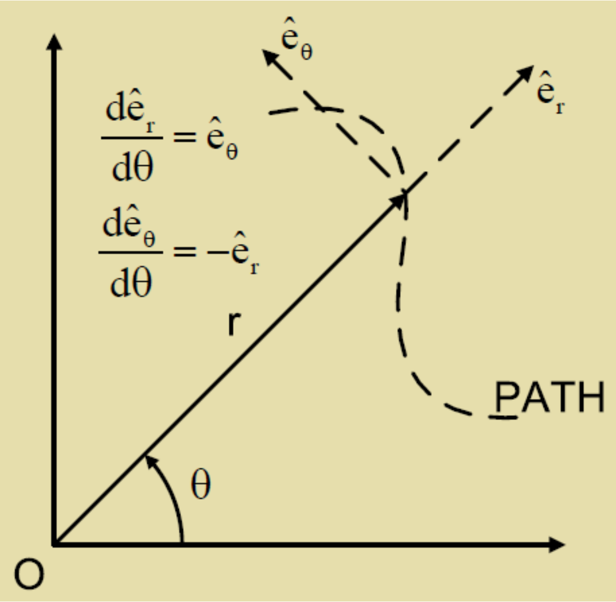
$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}sin\theta = -\omega_0^2 sin\theta$$

Derivation of Equations of Motion



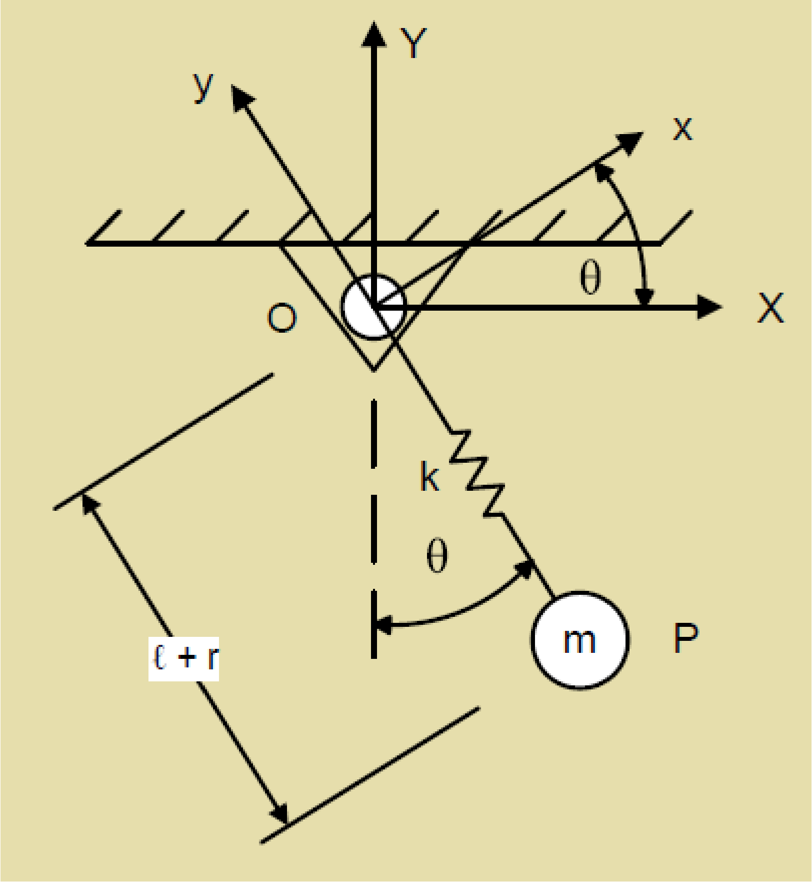
- m = pendulum mass
- m_{spring} = spring mass
- l = unstreatched spring length
- k = spring constant
- g = acceleration due to gravity
- F_t = pre-tension of spring
- r_s = static spring stretch, $r_s = \frac{mg - F_t}{k}$
- r_d = dynamic spring stretch
- r = total spring stretch $r_s + r_d$

Derivation of Equations of Motion -Polar Coordinates



- $\vec{r} = r\hat{e}_t$
- $\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\widehat{e_r} + r\dot{\theta}\widehat{e_\theta} = v_r\widehat{e_r} + v_\theta\widehat{e_\theta}$
- $\vec{a} = \frac{d\vec{v}}{dt} = \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{e_r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\widehat{e_\theta} + a_r\widehat{e_r} + a_\theta\widehat{e_\theta}$
- $v_r \begin{cases} \text{magnitude change} & \ddot{r} \\ \text{direction change} & \dot{r}\dot{\theta} \end{cases}$
- $v_\theta \begin{cases} \text{magnitude change} & r\ddot{\theta} + \dot{r}\dot{\theta} \\ \text{direction change} & r\dot{\theta}^2 \end{cases}$

Derivation of Equations of Motion
-Rigid Body Kinematics



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{bmatrix}$$

Derivation of Equations of Motion -Rigid Body Kinematics

$$ {}^{R}\vec{a}^{P} = {}^{R}\vec{a}^{O} + \left[{}^{R}\vec{\omega}^{R_1} \times \left({}^{R}\vec{\omega}^{R_1} \times \vec{r}^{OP}\right)\right] + \left[{}^{R}\vec{\alpha}^{R_1} \times \vec{r}^{OP}\right] + {}^{R_1}\vec{a}^{P} + 2\left[{}^{R}\vec{\omega}^{R_1} \times {}^{R_1}\vec{v}^{P}\right] $$

$$ {}^{R}\vec{a}^{O} = 0 $$

$$ {}^{R}\vec{\omega}^{R_1} = \dot{\theta}\hat{k} = \dot{\theta}\hat{K} $$

$$ \vec{r}^{OP} = -(\ell + r)\hat{j} = -(\ell + r)\left[-\sin\theta\hat{I} + \cos\theta\hat{J}\right] $$

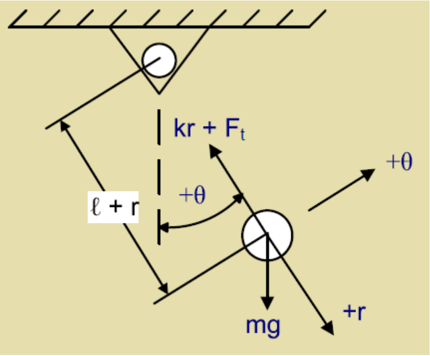
$$ {}^{R}\vec{\alpha}^{R_1} = \ddot{\theta}\hat{k} = \ddot{\theta}\hat{K} $$

$$ {}^{R_1}\vec{v}^{P} = -\dot{r}\hat{j} = -\dot{r}\left[-\sin\theta\hat{I} + \cos\theta\hat{J}\right] $$

$$ {}^{R_1}\vec{a}^{P} = -\ddot{r}\hat{j} = -\ddot{r}\left[-\sin\theta\hat{I} + \cos\theta\hat{J}\right] $$

After substitutions and evaluation:

$$ {}^{R}\vec{a}^{P} = \hat{i}\left[(\ell + r)\ddot{\theta} + 2\dot{r}\dot{\theta}\right] + \hat{j}\left[-\ddot{r} + (\ell + r)\dot{\theta}^2\right] $$



Free Body Diagram

$$ \sum F_r = ma_r = m\left[\ddot{r} - (\ell + r)\dot{\theta}^2\right] $$
$$ \sum F_\theta = ma_\theta = m\left[(\ell + r)\ddot{\theta} + 2\dot{r}\dot{\theta}\right] $$

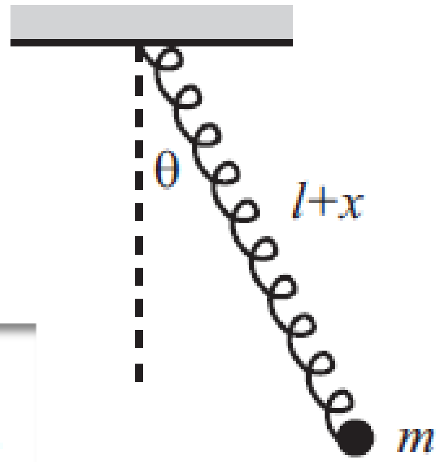
$$ m\ddot{r} - m(\ell + r)\dot{\theta}^2 + kr + F_t - mg\cos\theta = 0 $$
$$ (\ell + r)\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = 0 $$

Derivation of Equations of Motion -Lagrange Equations

$$T = \frac{1}{2}m\left(\dot{x}^2 + (\ell + x)^2\dot{\theta}^2\right).$$ Kinetic Energy

$$V(x,\theta) = -mg(\ell + x)\cos\theta + \frac{1}{2}kx^2.$$ Potential Energy

$$L \equiv T - V = \frac{1}{2}m\left(\dot{x}^2 + (\ell + x)^2\dot{\theta}^2\right) + mg(\ell + x)\cos\theta - \frac{1}{2}kx^2.$$



Derivation of Equations of Motion -Lagrange Equations

- Lagrange's Equation, Nonlinear equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \qquad \Longrightarrow \qquad m\ddot{x} = m(\ell + x)\dot{\theta}^2 + mg\cos\theta - kx,$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \qquad \Longrightarrow \qquad \frac{d}{dt}\left(m(\ell + x)^2\dot{\theta}\right) = -mg(\ell + x)\sin\theta$$

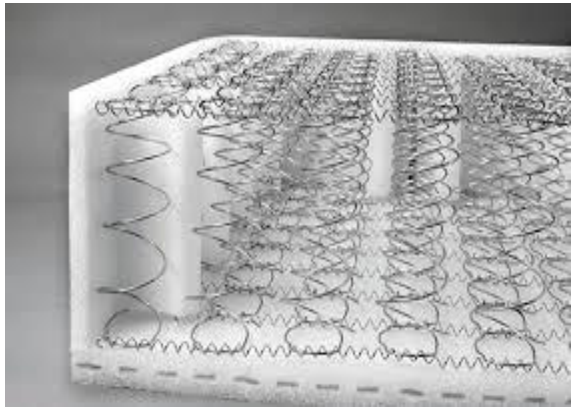
$$\Longrightarrow \qquad m(\ell + x)^2\ddot{\theta} + 2m(\ell + x)\dot{x}\dot{\theta} = -mg(\ell + x)\sin\theta.$$

$$\Longrightarrow \qquad m(\ell + x)\ddot{\theta} + 2m\dot{x}\dot{\theta} = -mg\sin\theta.$$

Elastic pendulum in the real world

Pendulum… but not elastic:

Elastic… but not pendulum:



Elastic pendulum in the real world – Spring Swinging

Fig. 1.

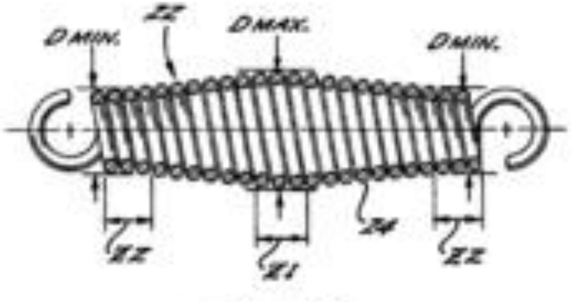
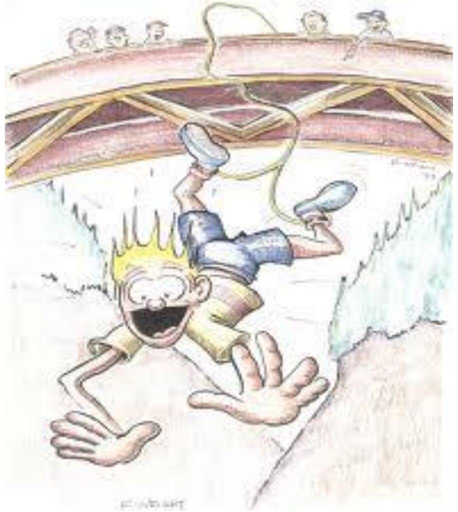


Fig. 2.

Elastic pendulum in the real world
-Bungee Jumping



Trivial Cases

- System not integrable
- Initial condition without elastic potential
- Only vertical oscillation
- Initial condition with elastic potential

Equilibrium States

- Hook's Law $F_e = -k(l - l_0)$
- Gravitational Force $F_g = mg$
- At equilibrium $\sum F = F_e + F_g = 0$
  $k(l - l_0) = mg$
- System at equilibrium

$$\ddot{x} = 0 \Rightarrow x = 0$$

$$\ddot{y} = 0 \Rightarrow y = 0$$

$$\ddot{z} = 0 \Rightarrow -\frac{k}{m}\left(\frac{l - l_0}{l}\right)z - g = 0 \Rightarrow z = -l$$

Stable state $(x, y, z) = (0, 0, -l)$

Unstable at $(x, y, z) = (0, 0, 2l_0 - l)$

Turning things into a handy system:

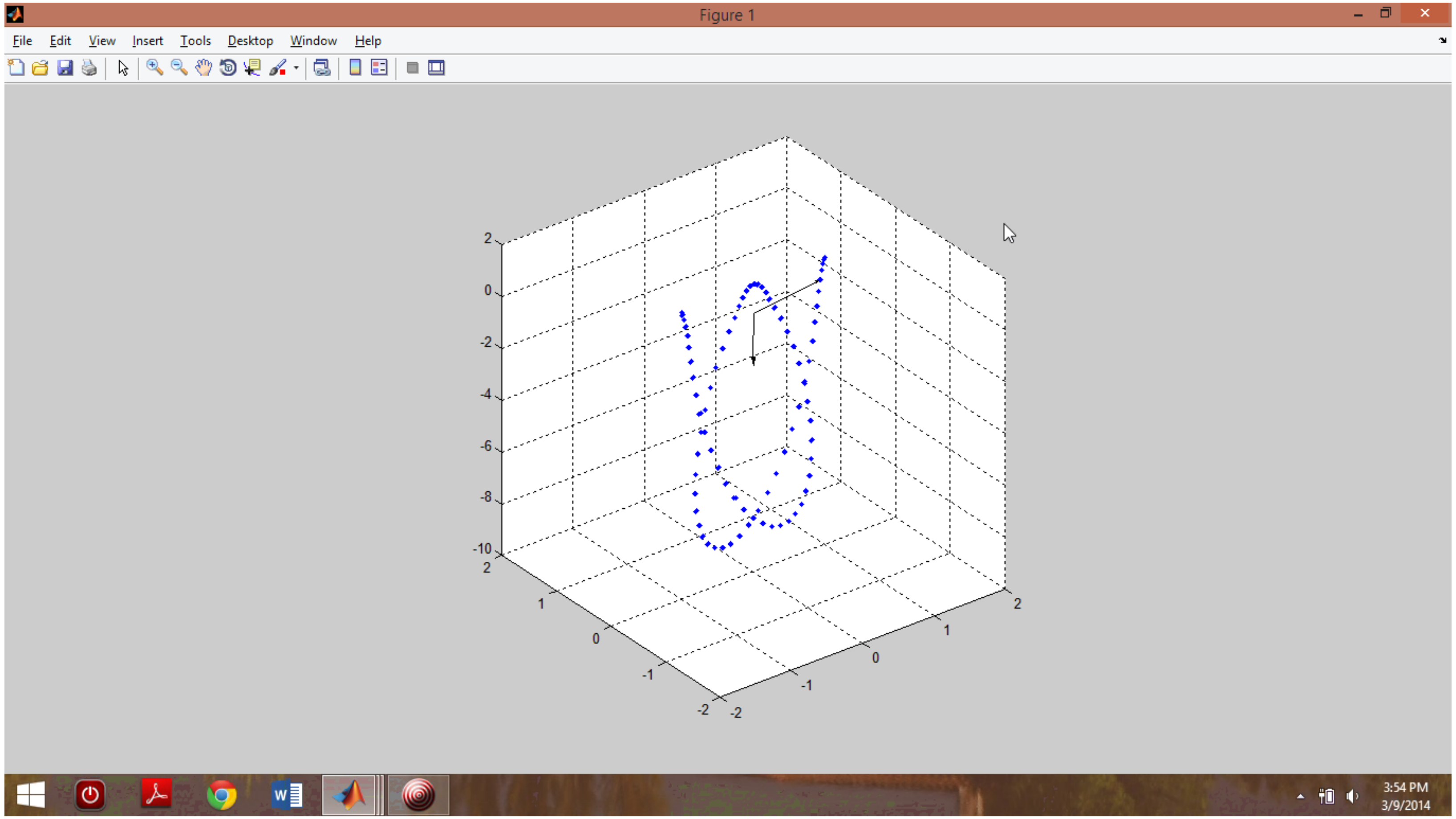
$$\ddot{x} = -\frac{\omega_z^2(r - l_0)}{r}x$$

$$\ddot{y} = -\frac{\omega_z^2(r - l_0)}{r}y$$

$$\ddot{z} = -\frac{\omega_z^2(r - l_0)}{r}z - g$$

Where $\omega_z = \sqrt{\frac{k}{m}}$ and $r = \sqrt{x^2 + y^2 + z^2}$

Some regimes make familiar shapes!



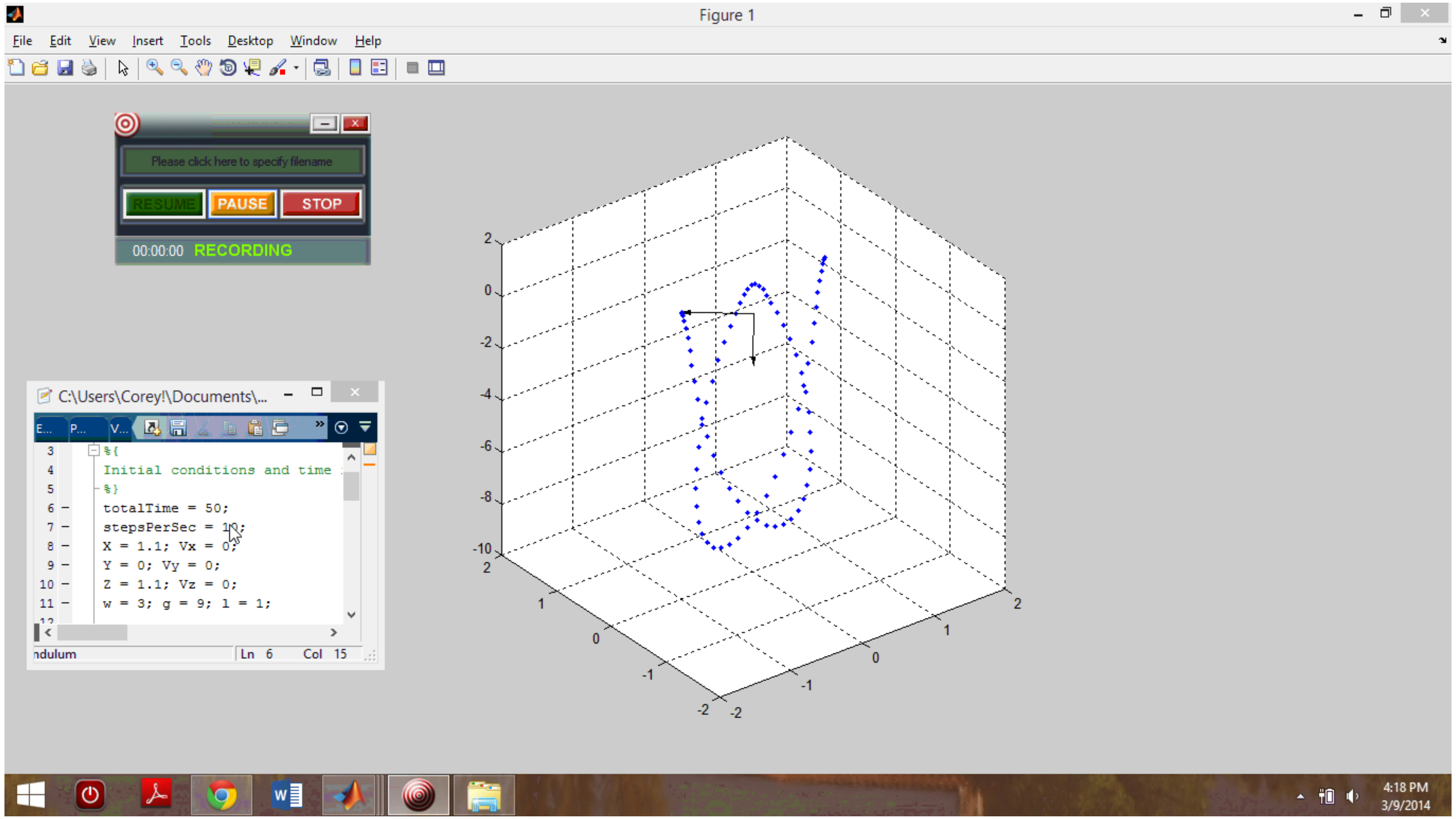
Initial Conditions:
X = 1; Vx = 0;
Y = 0; Vy = 0;
Z = 1.1; Vz = 0;

Parameters:
w = 3; g = 9; l = 1;

So motion stays in the XZ plane
Positive Z is vertical.

Motion is shown to be relatively changeless over 50s

What happens if we shake things up?



Initial Conditions

X = 1.1; Vx = 0;

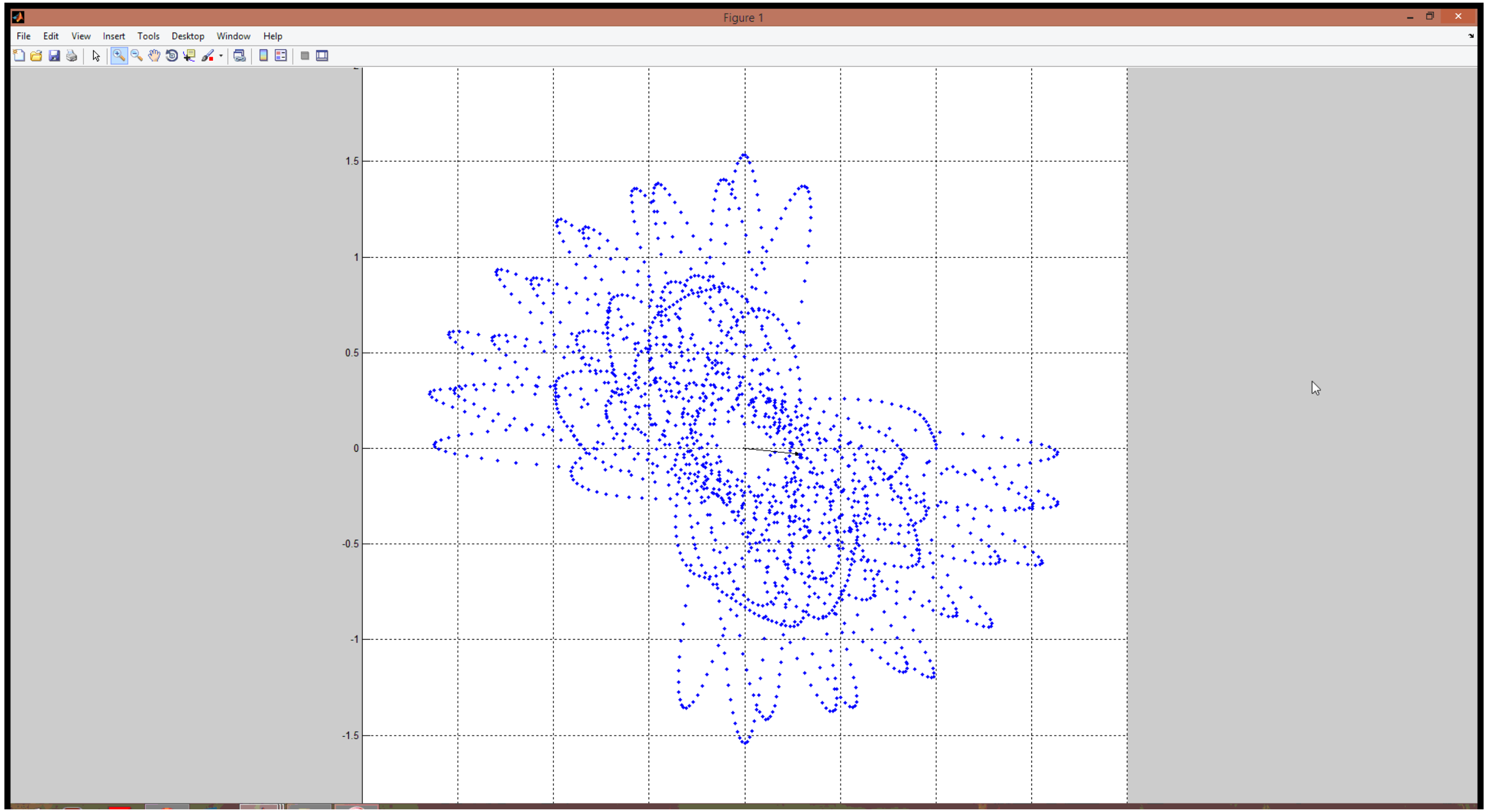
Y = 0; Vy = 0;

Z = 1.1; Vz = 0;

Parameters:

w = 3; g = 9; l = 1;

Turning things up a bit…

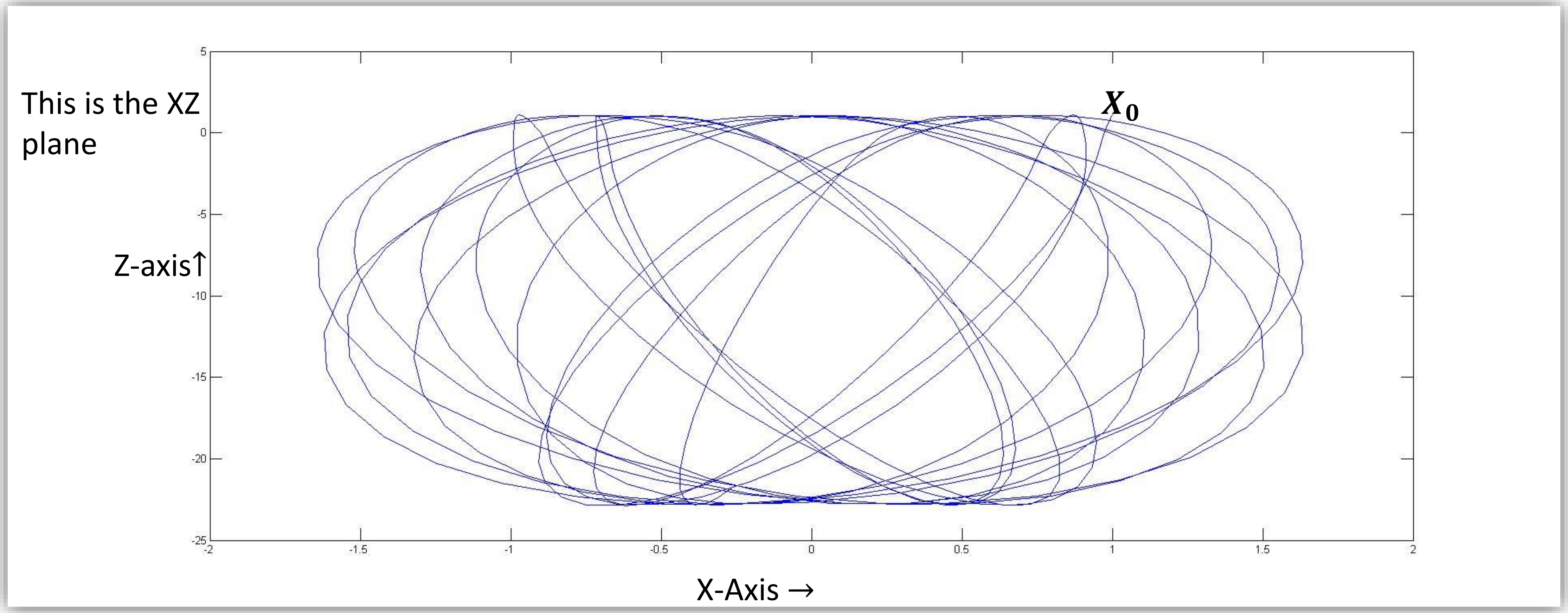


Initial Conditions:
X = 1; Vx = 0;
Y = 0; Vy = .2;
Z = 1.1; Vz = 0;

Parameters
w = 1; g = 10; l = 1;

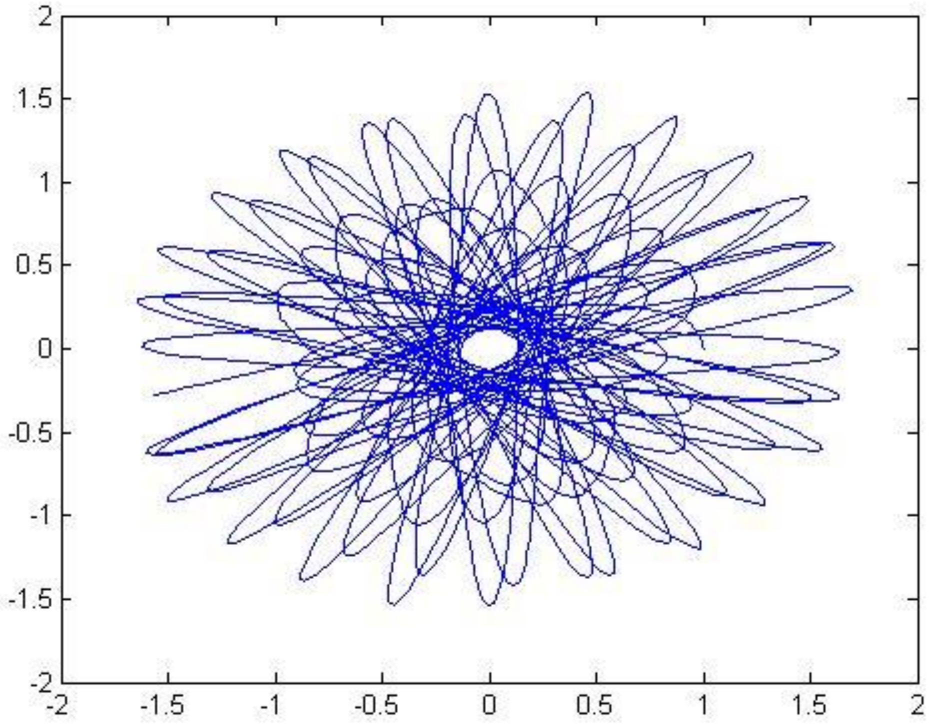
totalTime = 200;
stepsPerSec = 10;

Let's take a closer look at the same regime:

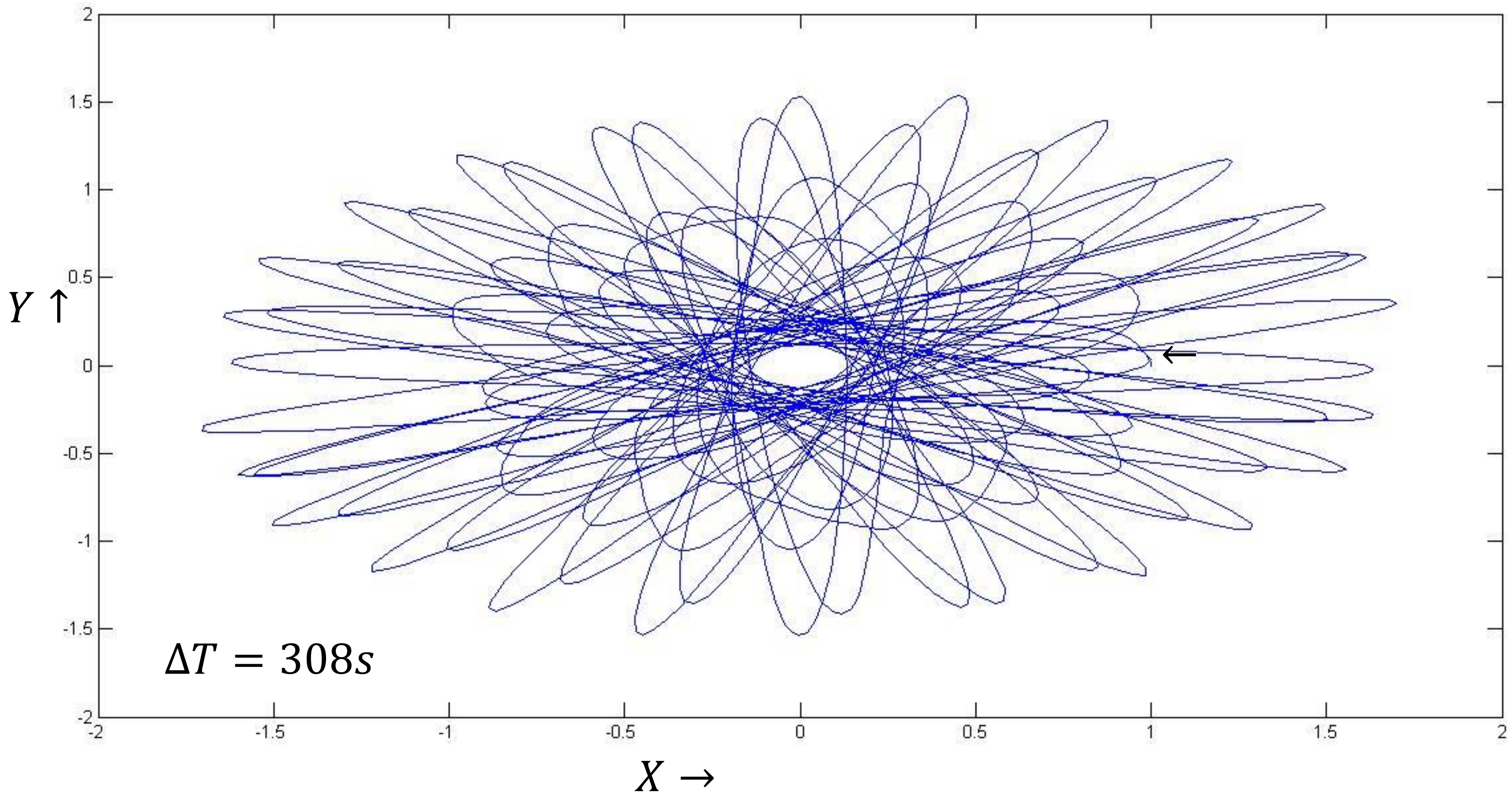


Now look at the XY plane again.

Is it the swivel that is causing the pendulum to avoid the center?



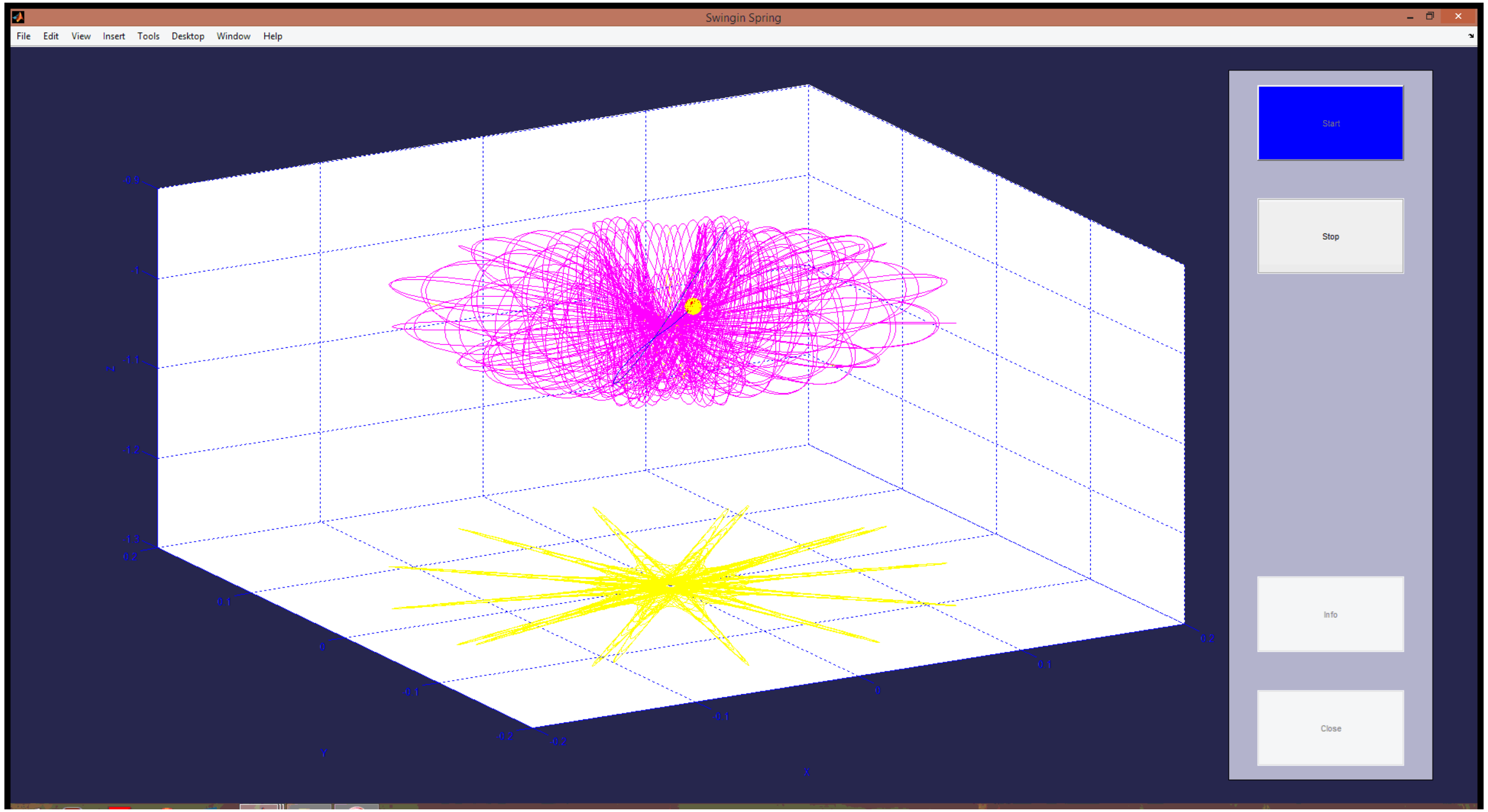
Awesome they almost meet! (Quasi-awesome)



Dr. Peter Lynch’s model:

Initial Conditions:

x0=0.01;
xdot0=0.00;
y0=0.00;
ydot0=0.02;
zprime0=0.1;
zdot0=0.00;



References (1/2)

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- Special thanks to Dr. Peter Lynch of the University College Dublin, Director of the UCD Meteorology & Climate Centre, for emailing his M-file and allowing us to include video of it's display of the fast oscillations of the dynamic pendulum!
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