ARIZONA

Hanging Chain Vibration Modes

MATHEMATICAL MODELING MATH 485-585

Project Description

- Hanging chain problems have been long been the a source of attention in Math and Physics as both didactic tools and research objects, even since the time of James Bernoulli (1655-1705) [1].
- Heavy, hang chains subject to vibration assume shapes of minimal bending stress and are subject to variable tension proportional to mass of chain below.
- Assuming small in-plane oscillation, it is well known that the classical solution oscillates sinusoidally in time with spatial profiles of zeroth order Bessel functions of the first kind [4].
- We set out to verify the classical solution with numerical and experimental methods.

Scientific Challenges

- Analytical solutions available only for small oscillation assumptions.

Potential Applications

- Modes of a rotating chain have been used for vertical axis wind turbines, such as the Darrieus turbine [2].

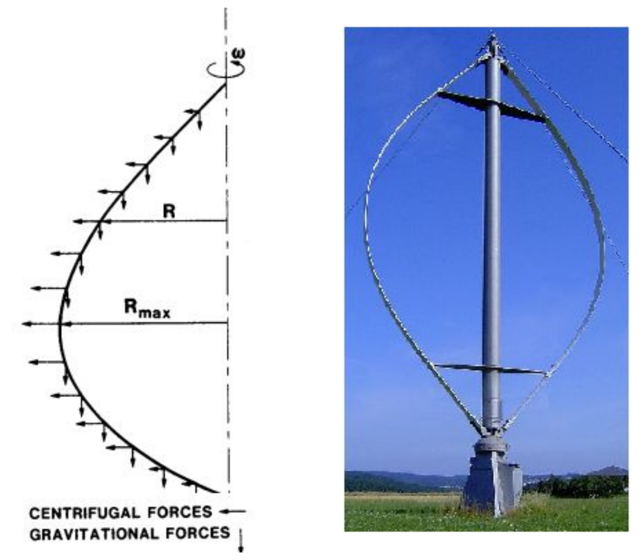

Figure 1: Improved Turbine (left) [2], Darrieus Turbine (right) [3]

Methodology

- Physical Experiment
  - Slow motion video of in-plane oscillations and rotating chain.
  - Conducted with metal chain and plastic beads of equal length.

Figure 2: Materials

N-Pendulum (Numerical)

Linear Model, Small Oscillations in terms of displacement U

$$U_{tt} = AU \qquad \mathbf{A} = n\begin{bmatrix} 1-2n & n-1 & & & \dots & 0 \\ n-1 & 3-2n & n-2 & & \dots & \\ & n-2 & 5-2n & n-3 & \dots & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ & & & 2 & -3 & 1 \\ 0 & & & & 1 & -1 \end{bmatrix}$$

Nonlinear Model, Matrix Equation in terms of the angle

$$\mathbf{M}(\theta)\theta_{tt} + \mathbf{C}(\theta)\theta_t^2 + \mathbf{g}(\theta) = 0$$

$$\mathbf{C}_{ij} = \frac{L}{N}(N - max(i,h))\sin(\theta_i - \theta_j)$$

$$\mathbf{M}_{ij} = \frac{L}{N}(N - max(i,h))\cos(\theta_i - \theta_j)$$

$$\mathbf{g}_i = g(N-i)\sin(\theta_i)$$

Team Members

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Methodology (Cont.)

Analytic Solution, Small in-plane dimensionless oscillation governing equation,

$$u_{tt} = u_x + x u_{xx}, \qquad u(1,t) \equiv 0$$

Separation of Variables solution,

$$u(x,t) = \sum_{n=0}^{\infty}\left\{A_n\sin(\lambda_n t) + B_n cos(\lambda_n t)\right\}J_0(2\lambda_n\sqrt{x})$$

where J is the zeroth order Bessel function of the first kind and natural frequencies, found by enforcing the B.C., $J_0(2\lambda_n) = 0$. Coefficients A, B are determined by the Fourier-Bessel series.

Results

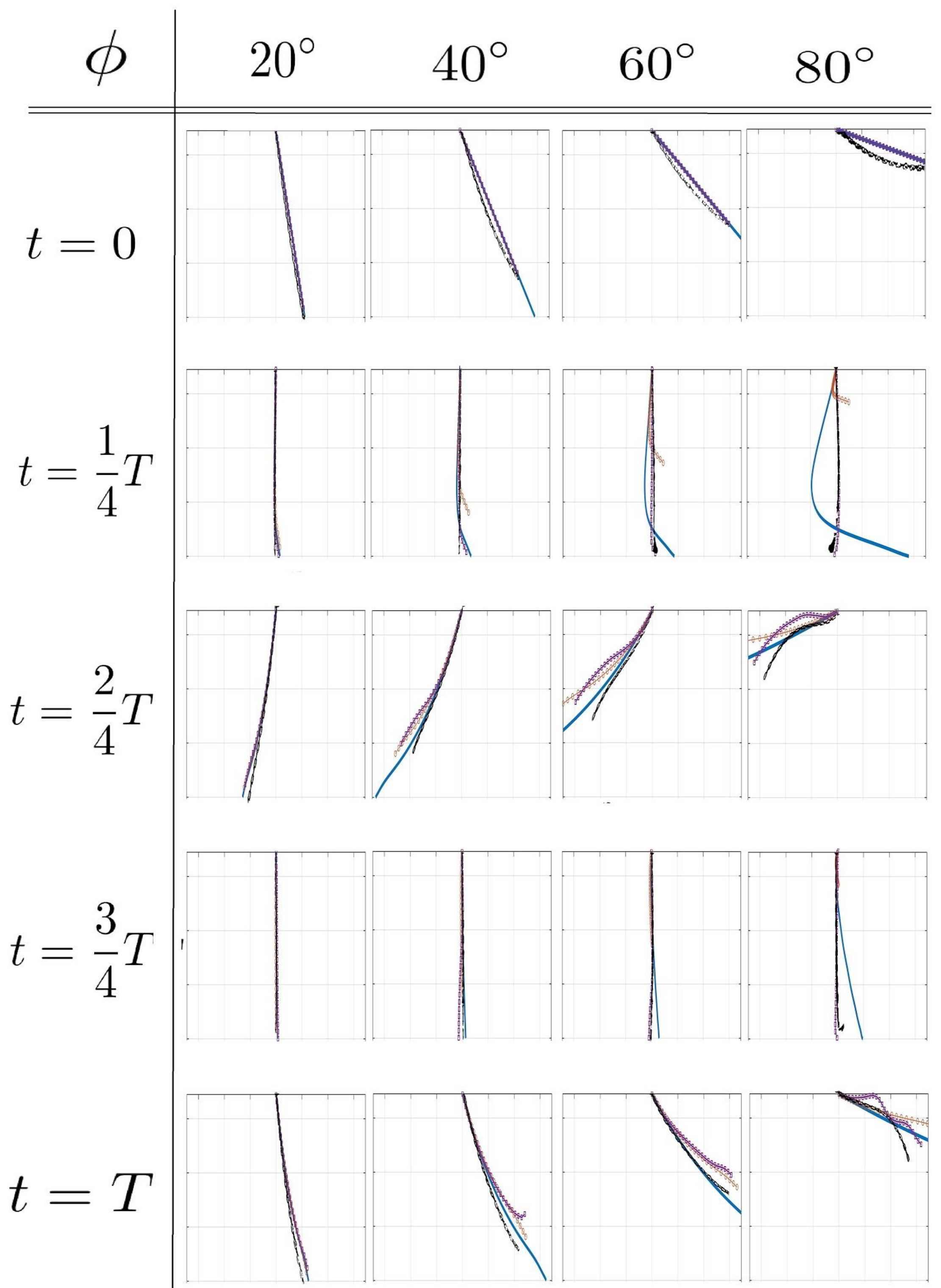

Figure 3: Experimental vs Numerical vs Analytic Results

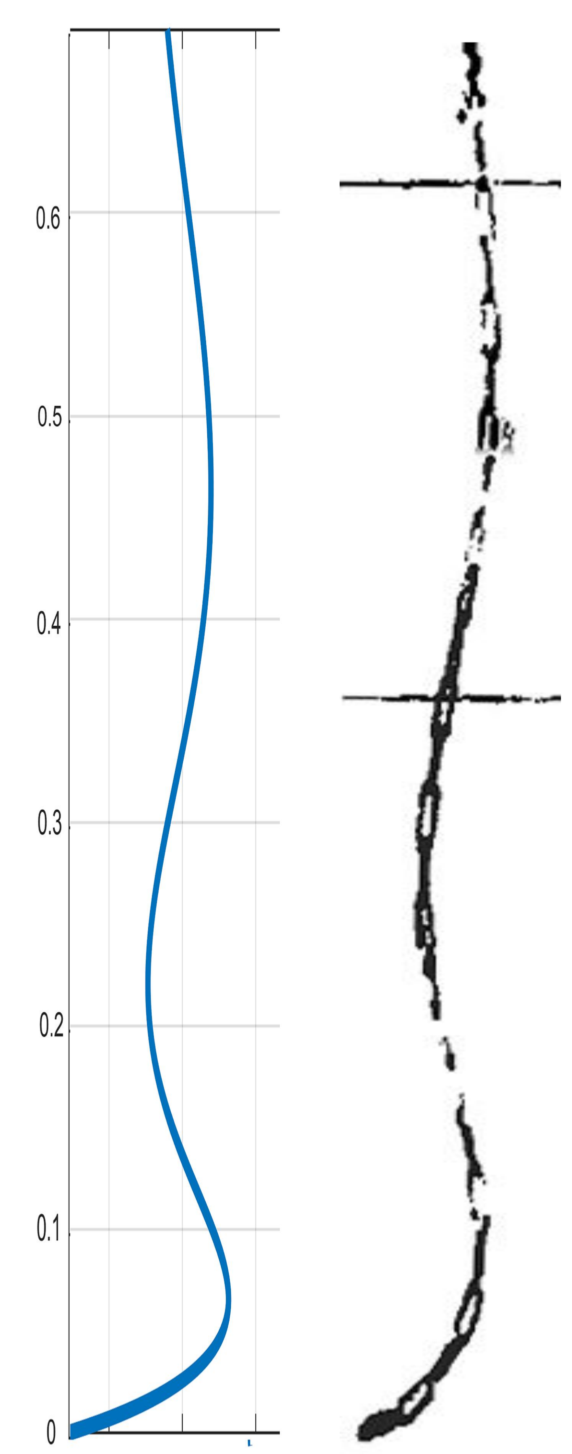

Figure 4: 4th Mode of Bessel solution (left), rotating chain (right)

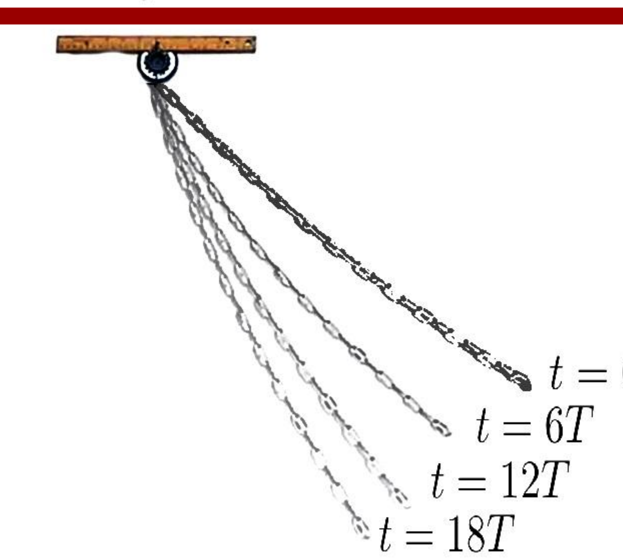

Figure 5: Air Drag, Max Amplitude vs t

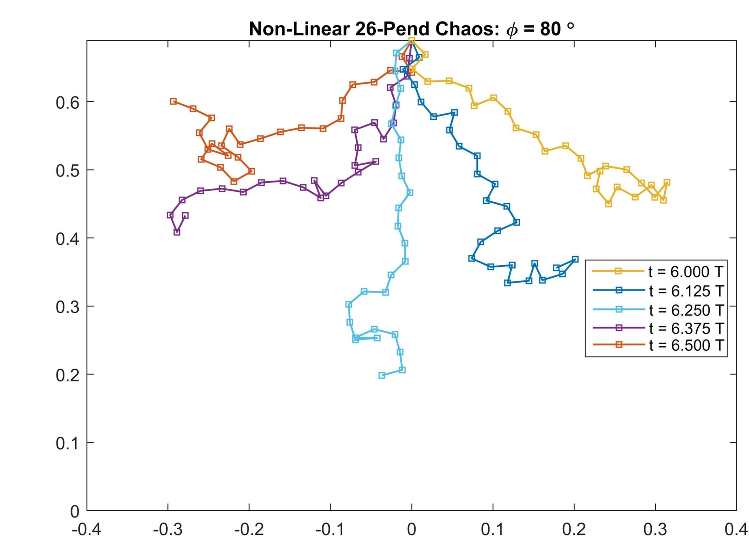

Figure 6: Non-linear 26-Pendulum Chaos

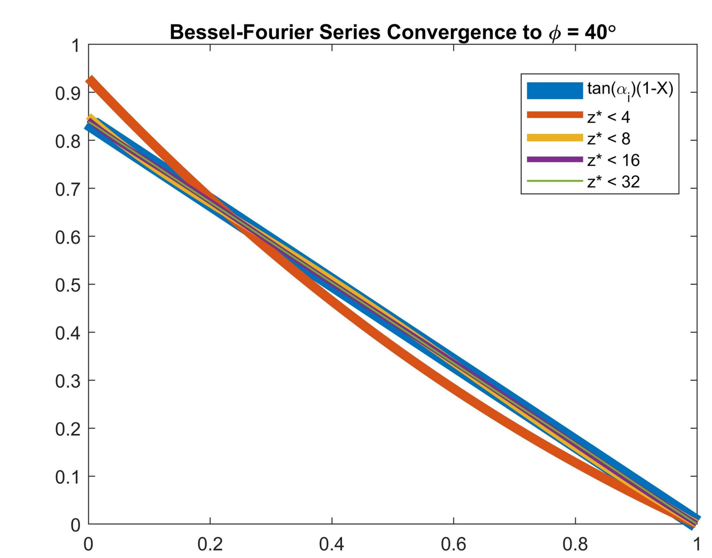

Figure 7: Initial Condition, Bessel-Fourier Series Convergence

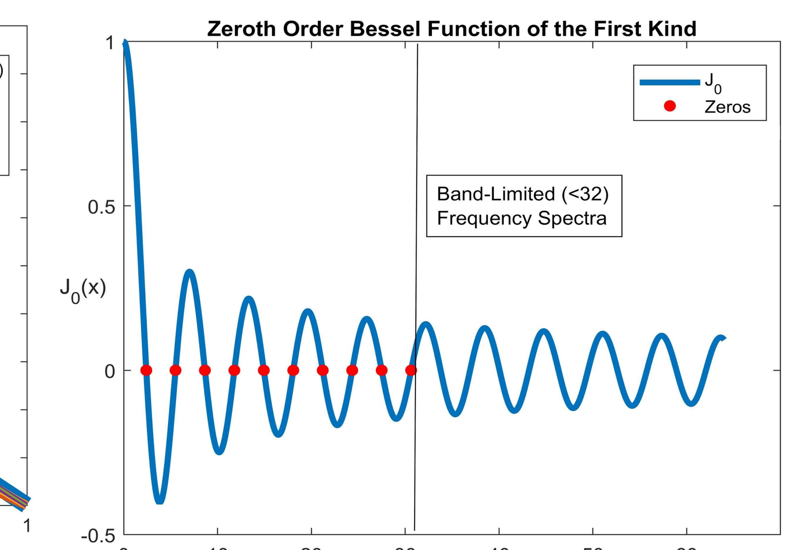

Figure 8: Zeroth Order Bessel Function of the First Kind

Glossary of Technical Terms

Fourier-Bessel Series of a function f is a projection onto a Bessel function basis $f(x) \sim \sum_{n=1}^{\infty} c_n J_\alpha$ with inner product as,

$$\langle f, g\rangle = \int_0^b x f(x) g(x)\,\mathrm{d}x \qquad c_n = \frac{\langle f, (J_\alpha)_n\rangle}{\langle (J_\alpha)_n, (J_\alpha)_n\rangle}$$

Zeroth Order Bessel Function of the First Kind: A power series solution $J_0(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}$,

to the zeroth order Bessel differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 y = 0$$

Acknowledgments

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References

1. Wilson, Edwin Bidwell. *The equilibrium of a heavy homogeneous chain in a uniformly rotating plane.* Annals of Mathematics **9** (3), 99–115 (1908).
2. Ashwill, Thomas D. *Developments in blade shape design for a darrieus vertical axis wind turbine.* Sandia Laboratory (1986).
3. Wacker, W. *A darrieus wind turbine has blades that approximate the shape of a troposkein to minimize bending stresses* (2005).
4. Yong, Darryl. *Strings, chains, and ropes.* SIAM Review **48** (4), arXiv: https://doi.org/10.1137/050641910 (2006).