Hanging Chain Vibration Modes

MATH 485: Mid-Term Presentation

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Outline

1. Model
   - Continuum Chain
   - N-Pendulum
2. Other Literature

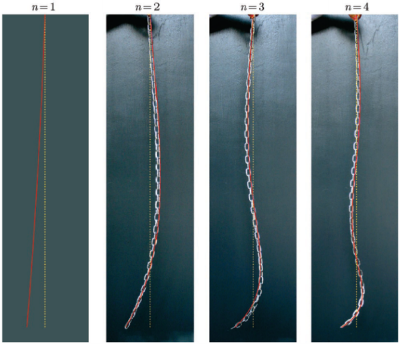

Figure: Hanging Chain [2]

Outline

1. Model
   - Continuum Chain
   - N-Pendulum
2. Other Literature

Hanging Chain Classical Model

Governing Equation

Model, $u(x, t)$

- u Transverse displacement $[m]$
- x Vertical height $[m]$
- t Time $[s]$
- ν Linear Density $[\frac{kg}{m}]$
- g Gravitational Acceleration $[\frac{m}{s^2}]$

Boundary Condition

- $u(L, t) \equiv 0$

Assumptions

- Small Oscillations.
- Uniform Density.

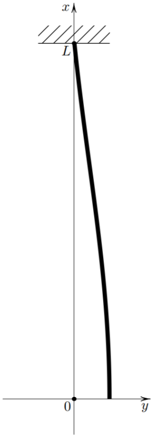

Figure: [1]

Hanging Chain Classical Model

Governing Equation

Observation

$\mathbf{T}(x)\cdot\hat{y} = \nu g x \tan(\theta) \approx \nu g x \frac{\partial u}{\partial x}$

Apply Newton's Third Law, $\sum \mathbf{F} \cdot \hat{y} = m\mathbf{a} \cdot \hat{y}$

$$\lim_{\Delta x \to 0} \frac{T(x+\Delta x) - T(x)}{\Delta x} = \nu \Delta x \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial}{\partial x}\left[\nu g x \frac{\partial u}{\partial x}\right] = \nu \frac{\partial^2 u}{\partial t^2}$$

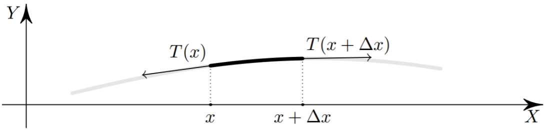

Figure: Hanging Chain [1].

Hanging Chain

Governing Equation

Governing Equation

$$u_{tt} = g(u_x + xu_{xx})$$

Let $X = \frac{x}{L}$, $U = \frac{u}{L}$, $\tau = \frac{t}{\sqrt{\frac{g}{L}}}$.

Thus, $X \in [0, 1]$, $U \in [0, 1]$, $\sqrt{\frac{g}{L}}$ characteristic time

Non-Dimensionalized Governing Equation

$$U_{\tau\tau} = U_X + XU_{XX} \qquad U(1, \tau) \equiv 0$$

Hanging Chain

Classical Solution

Governing Equation*

$U_{tt} = U_x + xU_{xx} \qquad u(1,t) \equiv 0$

Apply Separation of Variables. Let $U(X,\tau) = X(x) \cdot T(t)$, thus

$$\frac{T''}{T} = \frac{X' + x \cdot X''}{X} = -\lambda^2 \qquad \lambda \in \mathbb{R}$$

Let $z^2 = 4x$, use **chain** rule,

$$\begin{cases} \mathrm{T}'' + \lambda^2 T = 0 \\ z^2 X'' + zX' + z^2\lambda^2 X = 0 \end{cases}$$

Hanging Chain

Classical Solution

Governing Equations*

$$\begin{cases} T'' + \lambda^2 T = 0 \\ z^2 X'' + zX' + z^2\lambda^2 X = 0 \end{cases}$$

Time Oscillation $\begin{cases} \sin(|\lambda| t) \\ \cos(|\lambda| t) \end{cases}$

Spacial ODE is well known, Bessel Equation of zeroth order with solutions of Bessel function of the first and second kind

.

Spacial Profile, $\begin{cases} \mathrm{J}_0(\lambda z) = J_0(2\lambda\sqrt{x}) \\ \mathrm{Y}_0(\lambda z) = Y_0(2\lambda\sqrt{x})) \end{cases}$

Hanging Chain

Bessel Function

$Y_0(x)$ has an asymptote at $x = 0$.

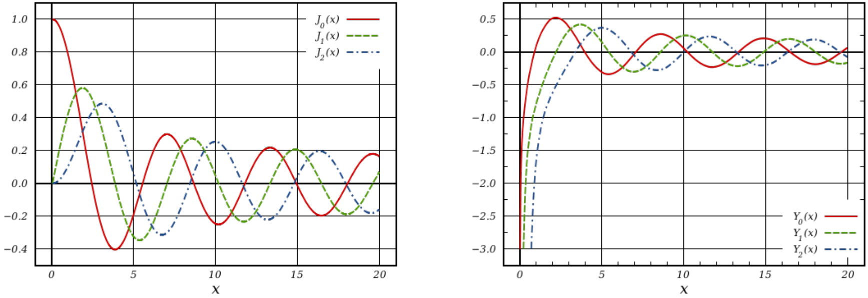

Figure: $J_0(x)$, $Y_0(x)$ [3]

Hanging Chain

Superposition

By linearity, $u(x,t) = X(x) \cdot T(t)$ is,

$$u(x,t) = \sum_{n=0}^{\infty} \left\{ A_n \sin(\lambda_n t) + B_n cos(\lambda_n t) \right\} J_0(2\lambda_n \sqrt{x})$$

Recall the boundary condition, $u(1,t) = 0$,

$$\sum_{n=0}^{\infty} \left\{ A_n \sin(\lambda_n t) + B_n cos(\lambda_n t) \right\} J_0(2\lambda_n \sqrt{x}) = 0$$

For non-trivial result, A_n,, $B_n \neq 0$, thus $J_0(2\lambda_n) = 0$, e.g. roots of J_0 determine λ_n.

Harmonic Frequencies

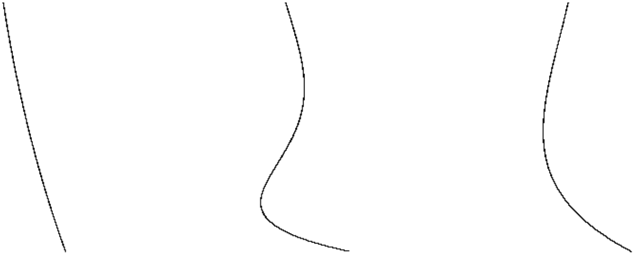

Figure: Modes 1-3, first three Bessel zeros [1]

Outline

N-Pendulum

Discrete Model, $u_i(t)$.

- u Transverse displacement $[m]$
- t Time $[s]$
- g Gravitational Acceleration $[\frac{m}{s^2}]$

Assumptions

- Small Oscillations.
- Equal
  Length and Mass, e.g. $m_i = m_j \quad l_q = l_p \forall i, j, p, q$

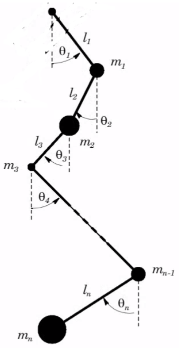

Figure: N-Pendulum [4].

N-Pendulum

Governing Equations

$\sum \mathbf{F} = m\mathbf{a}$, for small oscillations,
motion in y direction is negligible, and $\tan(\theta) \approx \theta$

$$F_i - F_{i-1} = m_i \ddot{u}_i$$

$$\sum_{j=i+1}^{n} m_j g tan(\theta_i) - \sum_{j=i}^{n} m_j g tan(\theta_{i-1}) = m_i \ddot{u}_i$$

$$(n-i)g\theta_i - (n-i+1)g\theta_{i-1} = \ddot{x}$$

Using the small angle approximation, $\theta_i \approx \frac{u_i}{l}$, we find,

$$\ddot{u}_i = \frac{g}{l}[(n-i)(u_{i+1} - 2u_i + u_{i-1}) - u_i + u_i]$$

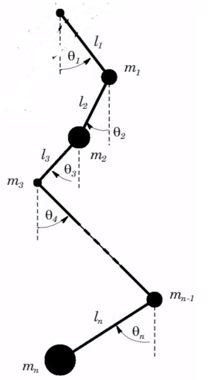

Figure: N-Pendulum [4].

N-Pendulum

Governing Equations

N-Pendulum Governing Equation

$$\ddot{U}_i = \frac{g}{l}[(n-i)(u_{i+1} - 2u_i + u_{i-1}) - u_i + u_i]$$

Let $U = \frac{u}{nL}$, $\tau = \frac{t}{\sqrt{\frac{g}{L}}}$.

Thus $U \in [0,1]$, $\sqrt{\frac{g}{L}}$ characteristic time

N-Pendulum Governing Equation*

$$U_{\tau\tau} = (n-i)(U_{i+1} - 2U_i + U_{i-1}) - U_i + U_i$$

N-Pendulum

Harmonic Frequencies

N-Pendulum Governing Equation**

$$\ddot{U} = (n-i)(U_{i+1} - 2U_i + U_{i-1}) - U_i + U_i$$

In matrix form, $\frac{d^2}{dt^2}\mathbf{U} = \mathbf{AU}$, where, $\mathbf{A}$ is given as,

$$\mathbf{A} = \begin{bmatrix} 1-2n & n-1 & & & \dots & \\ n-1 & 3-2n & n-2 & & \dots & \\ & n-2 & 5-2n & n-3 & \dots & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ & & & 2 & -3 & 1 \\ & & & & 1 & -1 \end{bmatrix}$$

N-Pendulum

Harmonic Frequencies

N-Pendulum Matrix Equation

$\frac{d^2}{dt^2}\mathbf{U} = \mathbf{A}\mathbf{U}$

Let $\mathbf{U}$ take a resonant form of $\mathbf{U} = \mathbf{v}e^{\omega_i t}$. Thus, $\frac{d^2}{dt^2}\mathbf{U} = \omega_i^2\mathbf{U}$. Thus,

$$(\mathbf{A} - \omega_i^2\mathbb{1})\mathbf{U} = \mathbf{0}$$

For non-trivial solution, we solve,

$$\det(\mathbf{A} - \omega_i^2\mathbb{1}) = 0$$

N-Pendulum

https://www.youtube.com/watch?v=XVIi7-u9wI4 [10]
https://youtu.be/-ztU56yG6aY
https://youtu.be/4FrdYwOIbjw

N-Pendulum

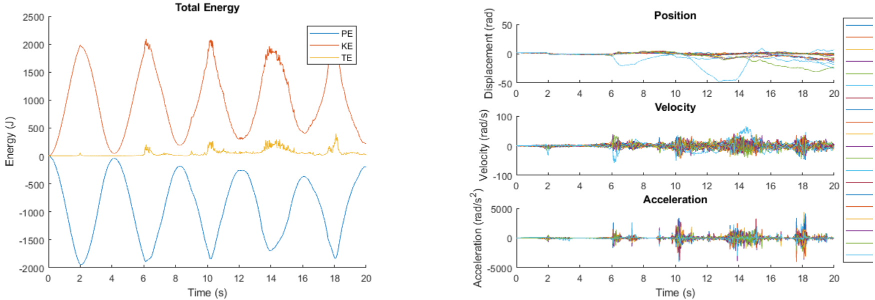

Figure: N=20 Pendulum Simulation [10]

Other Literature

- Concentrated tip-mass
  - $T(x) = mg + gx$, same equation up to translation
  - New Boundary Conditions
- Control Systems
  - Flatness
  - Spring Feedback

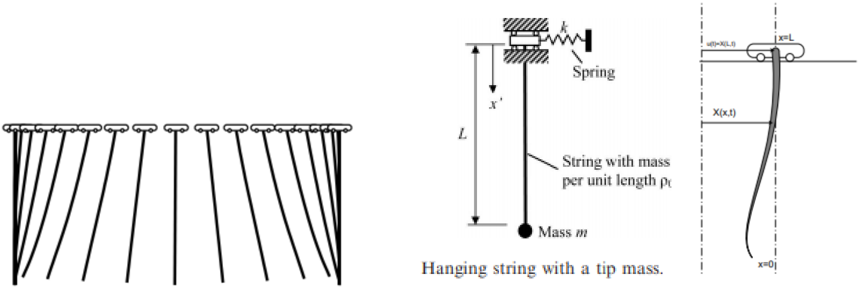

Figure: Modified Problems [10][?] [11]

Summary

- The classical hanging chain results in zeroth order first-kind Bessel function, J_0, standing waves.
- Natural frequencies are determined by zeroes of J_0.
- The $\lim_{n\to\infty}$ n-pendulum approaches similar behavior.
- Outlook
  - Verify Model
    - Numerically, Experimentally, Analytically
  - Introduce Complications
    - Air Drag
    - Non-uniform Density
    - Driven System (water hose)
    - Sliding Chain
    - Control Theory Stabilization
    - Magnetic Chains [6]

For Further Reading I

- Daniel A. Russel
  acs.psu.edu/drussell/Demos/HangChain/HangChain.html
- Daniel B. Friedman
  sjs.org/friedman/PDE/Examples/Hanging chain.pdf
- Charles Byrne
  sjs.org/friedman/PDE/course/bessel.pdf
- Steven P. Weibel John Baillieul (2010) Open-loop oscillatory stabilization of an n -pendulum, International Journal of Control, 71:5, 931-957, DOI: 10.1080/002071798221641
  https://www.tandfonline.com/doi/abs/10.1080/00207179822164
- S. Weibel ; J. Baillieul ; B. Lehman, "Equilibria and stability of an n-pendulum forced by rapid oscillations," Decision and Control, 1997
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For Further Reading II

- S. Weibel ; J. Baillieul ; B. Lehman, "Stability of vertical magnetic chains," The Royal Society, 2017
  http://rspa.royalsocietypublishing.org/content/473/2198/20
- S. Weibel, J. Baillieul and B. Lehman, "Equilibria and stability of an n-pendulum forced by rapid oscillations," Proceedings of the 36th IEEE Conference on Decision and Control, San Diego, CA, 1997, pp. 1147-1152 vol.2. doi: 10.1109/CDC.1997.657602
  http://ieeexplore.ieee.org/abstract/document/657602/?reloa
- fake_crayonphysics, srli, GitHub
  https://github.com/srli/fake_crayonphysics
- Boundary Conditions and Modes of the Vertically Hanging Chain Y. Verbin, 2014 arXiv:1412.1846 [physics.class-ph]
  https://arxiv.org/pdf/1412.1846.pdf

For Further Reading III

- C. Y. Wang C. M. Wang (2010) Exact solutions for vibration of a vertical heavy string with a tip mass, The IES Journal Part A: Civil Structural Engineering, 3:4, 278-281, DOI: 10.1080/19373260.2010.521623 https://www.tandfonline.com/doi/pdf/10.1080/19373260.2010.
- Flatness of Heavy Chain Systems, Nicolas Petit and Pierre Rouchon, SIAM 1995 https://doi.org/10.1137/S0363012900368636