Hanging Chain Vibration Modes
MATH 485 Mid-Term Report

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We develop models to describe small displacement, small relative to the total length of the chain L, of a hanging chain. We find sinusoidal time-dependent solutions and Bessel function spatial solutions. We propose to experimentally verify the model using a drill, chain, and high-speed (1000 fps) camera. The model neglects friction due to the interaction of fluid (air) with the chain, hence we propose to correct this by introducing friction terms into the model and experimentally verify damping coefficients. Moreover, we propose to experimentally and numerically consider the fluid-structure interaction between rotating heavy-chain immersed in more viscous fluid, water.

1. Background/Introduction

Hanging chain problems continue to attract attention in physics and mathematics since James Bernoulli first solved the case of a uniform chain hanging due to gravity [Wilson (1908)]. The classical solution involves special functions, eigenvalues, and separation of variables; thus, the problem is highly didactic of the methods involved and is presented in a number of classical mechanics textbooks [Morin (2003), McKay (2003)]. Furthermore, the hanging chain model, due to its accessibility and low cost, is easily experimentally verified by students and researchers alike. Moreover, the case on hanging chains is not closed and research on hanging chains is on-going today [Fritzkowski & Kaminski (2013), Verbin (2015), Petit & Rouchon (2001), Wang & Wang (2010)]. The hanging chain system presents an opportunity not only to contribute to novel research; but perhaps equally important, provides an opportunity contribute to the history of the problem's use in mathematical education. In section 2, we present the classical linearized hanging chain solution and develop motivation to study the non-linear hanging chain using the n-pendulum model. Lastly, we lay out our plan to (1) verify the model, (2) introduce friction, and (3) consider the chain immersed in water.

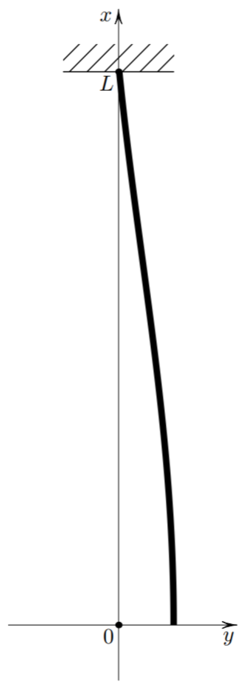

Figure 1: Hanging Chain (Rozman 2017).

2. Model

2.1. *Classical Derivation*

We develop the governing equation with newtonian physics. Fix the origin of our coordinate system at the un-fixed end of the chain (Figure 1). We consider displacement of the chain in the $\hat{\mathbf{x}}$ direction to be small relative to the length of the chain, L, and thus displacement, $\mathbf{u}$, is only considered in the in-plane $\hat{\mathbf{y}}$ direction and parameterized by vertical position x; and as such, we seek a model to describe u(x,t). Variables are defined as follows:

- u Transverse displacement $[m]$
- x Vertical height $[m]$
- t Time $[s]$
- ν Linear Density $[\frac{kg}{m}]$
- g Gravitational Acceleration $[\frac{m}{s^2}]$

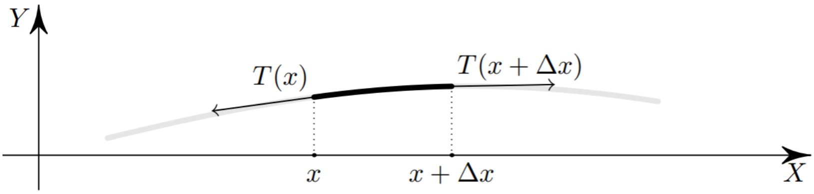

Figure 2: Hanging Chain.

Consider a small section of the chain of size Δx (Figure 2). We resolve forces in the $\hat{\mathbf{y}}$ direction, $\sum \mathbf{F} \cdot \hat{y} = m\mathbf{a} \cdot \hat{y}$, and take $\Delta x \to 0$ as follows,

$$\mathbf{T}(x + \Delta x) \cdot \hat{y} - \mathbf{T}(x) \cdot \hat{y} = \nu \Delta x \frac{\partial^2 u}{\partial t^2} \tag{2.1}$$

$$\lim_{\Delta x \to 0} \frac{\mathbf{T}(x + \Delta x) \cdot \hat{y} - \mathbf{T}(x) \cdot \hat{y}}{\Delta x} = \nu \frac{\partial^2 u}{\partial t^2} \tag{2.2}$$

$$\frac{\partial}{\partial x} \left\{ \mathbf{T}(x) \cdot \hat{y} \right\} = \nu \frac{\partial^2 u}{\partial t^2} \tag{2.3}$$

Observe that at an arbitrary point x in the chain, the vertical component of tension, $\mathbf{T}(x) \cdot \hat{x}$, must support the weight of the chain below it, $\nu g x$, thus $\mathbf{T}(x) \cdot \hat{x} = \nu g x$. Hence, we find $\mathbf{T}(x) \cdot \hat{y}$ by defining the angle between $\mathbf{T}$ and the $\mathbf{x}$ directions as θ. Further more, note that, since Tension is tangent to the chain, $\tan(\theta) = \frac{\partial u}{\partial x}$, thus, $\mathbf{T}(x) \cdot \hat{y} = \nu g x \frac{\partial u}{\partial x}$. Hence, returning to equation (2.3), we arrive at the governing equation,

$$\frac{\partial}{\partial x} \left[\nu g x \frac{\partial u}{\partial x} \right] = \nu \frac{\partial^2 u}{\partial t^2} \tag{2.4}$$

The chain is assumed to have uniform density and subject to a uniform gravitational field, thus g and ν are constants.

$$\frac{\partial^2 u}{\partial t^2} = g\left(\frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2}\right) \tag{2.5}$$

Next, we non-dimensionalize our equation. Let $X = \frac{x}{L}$, $U = \frac{u}{L}$, and $\tau = \frac{t}{\sqrt{\frac{g}{L}}}$. Thus, $X \in [0, 1]$, $U \in [0, 1]$, and $\sqrt{\frac{g}{L}}$ describes a characteristic time of the system, well known as the period of the simple pendulum. Substituting these variables yields the following non-dimensionalized governing equation and boundary condition

$$U_{\tau\tau} = U_X + X U_{XX}, \qquad U(1, \tau) \equiv 0. \tag{2.6}$$

For convenience, we switch back to conventional notation, with the understanding that we are working with the non-dimensionalized equation here-on-out unless otherwise stated, e.g.

$$u_{tt} = u_x + x u_{xx}, \qquad u(1, t) \equiv 0. \tag{2.7}$$

We now use the method of separation of variables. We say that the solution U takes the form $u(x, t) = X(x) \cdot T(t)$. Thus equation 2.7 reduces to the following.

$$\frac{T''}{T} = \frac{X' + x \cdot X''}{X} = -\lambda^2, \qquad \lambda \in \mathbb{R}. \tag{2.8}$$

In the standard way, the LHS and RHS are parameterized by different variables. The two are equal for all t and x, thus, both the LHS and RHS must equal some constant.

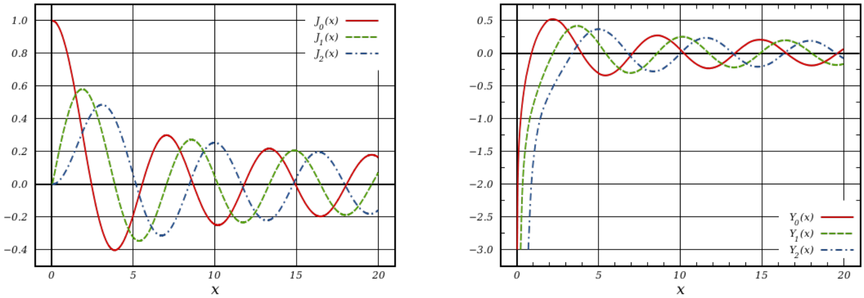

Figure 3: $J_0(x)$, $Y_0(x)$ [Byrne (2007)]

Moreover, with foresight, we recognize that the LHS and RHS must equal some negative constant to avoid exponential and linearly growing time-dependent solutions. As a result, we arrive at the decoupled system,

$$T'' + \lambda^2 T = 0 \tag{2.9}$$

$$xX'' + X' + x^2\lambda^2 X = 0 \tag{2.10}$$

Solutions to the eq (2.9) are well known and describe oscillations of the form $\sin(|\lambda|t)$, $\cos(|\lambda|t)$. Letting $z^2 = 4x$ and using **chain** rule reduces eq (2.10) to,

$$z^2X'' + zX' + z^2\lambda^2 X = 0 \tag{2.11}$$

Eq (2.11) is well known as a zeroth order Bessel equation with linearly independent power series solutions J_0, Y_0 known as the Bessel functions of the first and second kinds respectively. These solutions can be found with Method of Frobenius [Byrne (2007)]. Hence the spacial profile is given by,

$$J_0(\lambda z) = J_0(2\lambda\sqrt{x}) \tag{2.12}$$

$$Y_0(\lambda z) = Y_0(2\lambda\sqrt{x})). \tag{2.13}$$

$Y_0(x)$ has an asymptote at $x = 0$ (see figure 3) which corresponds with the end of the chain and thus is un-physical hence we discard the Y_0 solution. By linearity, and the superposition principle, we arrive at the general solution,

$$u(x,t) = \sum_{n=0}^{\infty} \left\{ A_n \sin(\lambda_n t) + B_n cos(\lambda_n t) \right\} J_0(2\lambda_n\sqrt{x}). \tag{2.14}$$

Recalling the boundary condition, $u(1,t) = 0$, we get

$$\sum_{n=0}^{\infty} \left\{ A_n \sin(\lambda_n t) + B_n cos(\lambda_n t) \right\} J_0(2\lambda_n) = 0 \tag{2.15}$$

$$J_0(2\lambda_n) = 0 \tag{2.16}$$

For non-trivial result, A_n, $B_n \neq 0$, thus roots of J_0 determine natural frequencies λ_n. In Figure 4, we present the first three mode shapes corresponding with the first three zeroes of the Bessel function.

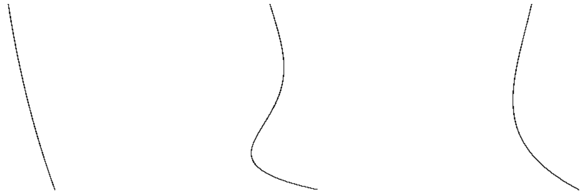

Figure 4: Vibration Mode Shapes (1-3, left to right) Russell (2011)

2.2. *N-Pendulum*

The above section 2.1 presents the derivation of the small displacement assumption governing equation of the chain modeled as a continuum. In order to consider the dynamics of discrete links in a chain, we consider the N-Pendulum model (Figure 5). For our purposes, we consider the length of each pendulum arm l_i to be uniform $l_i = \frac{L}{N}$ where L is the total length of all the pendulum arms and N is the number of arms. Further, we consider each mass m_i to be uniform $m_i = \frac{M}{N}$ where M is the total mass. Moreover, to gain insight into the non-linear behavior of the hanging chain problem, we consider the non-linear n-pendulum system as a bridge to numerically study the behavior of the hanging chain without small displacement assumptions [Fritzkowski & Kaminski (2013), Wang (1994)].

We seek a model of n-coupled differential equations to describe the in-plane displacement of each mass, m_i. We apply Newtonian mechanics and conclude that the force on mass m_i is due moving pivot about mass m_{i-1} and the weight of the $n-i$ pendulum masses below.

$$F_i - F_{i-1} = m_i \ddot{u}_i \tag{2.17}$$

In a similar manner to the analysis in section 2.1, we observe that the vertical component of tension, $\mathbf{T_i} \cdot \hat{x}$, must support the weight of the chain below it, $\sum_{j=i+1}^{n} m_j g$, thus

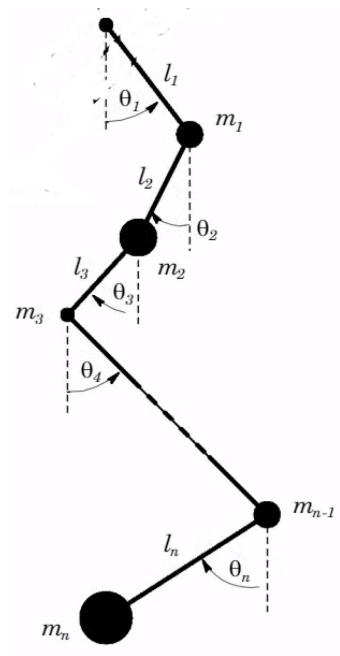

Figure 5: N-Pendulum Weibel & Baillieul (1998).

$$\mathbf{T_i} \cdot \hat{x} = \sum_{j=i+1}^{n} m_j g \tag{2.18}$$

Hence, we find $\mathbf{T}(x) \cdot \hat{y}$ with θ_i as,

$$\mathbf{T_i} \cdot \hat{y} = \tan(\theta_i) \sum_{j=i+1}^{n} m_j g \tag{2.19}$$

Hence, returning to equation (2.17), we arrive at the governing equation,

$$\sum_{j=i+1}^{n} m_j g \tan(\theta_i) - \sum_{j=i}^{n} m_j g \tan(\theta_{i-1}) = m_i \ddot{u}_i \tag{2.20}$$

Note that each mass m_i is connected by fixed length pendulum arms. Thus, the above equation (2.20) determines the displacement in the $\hat{x}$ and simultaneously determines displacement in the $\hat{y}$ direction. We plan to use equation (2.20) to model the non-linear hanging chain system Wang (1994). We can further linearize equation (2.20) using small oscillations and displacement assumptions, $\theta_i << 1$ and $\frac{u_i}{L} << 1$ respectively. We linearize the N-Pendulum system as follows. First, variables are defined as follows:

- u_i i-th Transverse Displacement in $\hat{y}$ direction $[m]$
- t Time $[s]$
- g Gravitational Acceleration $[\frac{m}{s^2}]$

Assumptions

- Small Oscillations (θ_i<<1) & Small Displacement ($\frac{u_i}{L}$<<1)
- Equal Length and Mass, e.g. $m_i = m_j \quad l_q = l_p \qquad \forall i, j, p, q \in [1, N]$

Using the small angle approximation, $\tan(\theta_i) \approx \sin(\theta_i) = \frac{u_{i+1}-u_i}{l_{i+1}}$, we find (2.20) becomes

$$\sum_{j=i+1}^{n} m_j g \frac{u_{i+1} - u_i}{l_{i+1}} - \sum_{j=i}^{n} m_j g \frac{u_i - u_{i-1}}{l_i} = m_i \ddot{u}_i \tag{2.21}$$

Assuming equal, uniform mass and length, where $m_i = \frac{M}{N}$, $l_i = \frac{L}{N}$

$$\ddot{u}_i = \frac{g}{L/N}[(N-i)(u_{i+1} - 2u_i + u_{i-1}) - u_i + u_{i-1}]$$

Let $U = \frac{u}{L/N}$, $\tau = \frac{t}{\sqrt{\frac{g}{L/N}}}$. Thus $U \in [0, 1]$ and $\sqrt{\frac{g}{L/N}}$ describes the characteristic time of the system, the same as we found in section (2.1). The non-dimension equation becomes,

$$U_{\tau\tau} = (n-i)(U_{i+1} - 2U_i + U_{i-1}) - U_i + U_{i-1}$$

In matrix form,

$$\frac{d^2}{d\tau^2}\mathbf{U} = \mathbf{AU} \tag{2.22}$$

where, $\mathbf{A}$ is given as,

$$\mathbf{A} = \begin{bmatrix} 1-2n & n-1 & & & \dots & 0 \\ n-1 & 3-2n & n-2 & & \dots & \\ & n-2 & 5-2n & n-3 & \dots & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ & & & 2 & -3 & 1 \\ 0 & & & & 1 & -1 \end{bmatrix} \tag{2.23}$$

The n-pendulum has been studied for unstable Inverted equilibria standing on its end and it has been found that the unstable equilbiria vanishes as $n \to \infty$ [Weibel & Baillieul (1998)]. Moreover, the natural frequencies approach zeroes of the Bessel function of the first kind. Hence, it appears the the N-Pendulum system does in-deed share characteristics with the hanging chain problem. We plan to numerically model both the linearized and nonlinearized N-Pendulum systems described by equations (2.22-23, 2.20 respectively). We have made some progress already on this leveraging an open source GitHub physics code repository [Li (2015)]. The code is not written very well (hard coded values, not modular, poor style, etc.) and will be written from scratch but provides us with a quick method to run simulations. Figure 6 presents results of an $N = 20$-Pendulum.

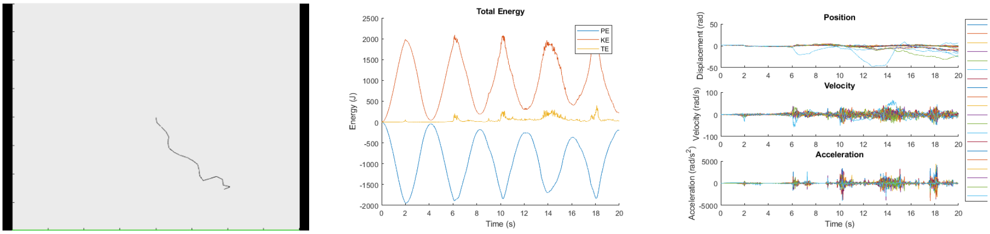

Figure 6: N=20 Pendulum Simulation [Source Code via Li (2015)]

3. Future Plan

Our plan for future work is three pronged,

(i) Verification
- Compare experimental vs. numerical vs. analytical results

(ii) Extension (Air Friction)
- Introduce air-friction drag term $-ku_t$ for some friction coefficient $k > 0 \in \mathbb{R}$.
- Verify the new model compare experimental vs. numerical vs. analytical results

(iii) Exploration (Water Friction)
- Introduce strong fluid-chain interaction by considering the rotating chain immersed in fluid (water).
- Perform experiment, and, as time permits, construct numerical model based on the Immersed Boundary Method

3.1. *Verification*

To verify our models experimentally, we plan to record in-plane hanging chain vibration modes with a high speed (1000 fps) with the Casio EX-FH25 High Speed Digital Camera. We plan to use both a chain, but we also plan to consider Mardi-Gras beads as a good approximation for an $N - Pendulum$ System.

We plan to verify the small displacement hanging chain model (Section 2.2) comparing numerical, analytical, and experimental results for the following cases,

(i) In-Plane Oscillations
(ii) Chain supporting a concentrated mass, M [Verbin (2015)]
(iii) Rotating Chain [Yong (2006)]

Items (ii) and (iii) in the above list are not explicitly mentioned in Section 2, but are known to have linearized analytic solutions of extremely similar forms to item (i). The concentrated mass introduces an extra constant term in the expression for tension in the chain [Verbin (2015)], resulting in the following equation,

$$\frac{\partial^2 u}{\partial t^2} = g(\frac{\partial u}{\partial x} + (x + M)\frac{\partial^2 u}{\partial x^2}) \tag{3.1}$$

The rotating chain governing equation is identical to eq (2.7), except parameterized by displacement from the vertical axis $\hat{x}$, x, and arc-length, s [Yong (2006)], as follows

$$x_{tt} = u_s + su_{ss}, \qquad u(L, t) \equiv 0. \tag{3.2}$$

with solution,

$$x(s,t) = \sum_{n=0}^{\infty} \left\{ A_n \sin(\lambda_n t\sqrt{\frac{g}{L}}) + B_n cos(\lambda_n t\sqrt{\frac{g}{L}}) \right\} J_0(2\lambda_n\sqrt{\frac{s}{L}}). \tag{3.3}$$

Numerical simulations will use the n-pendulum as an approximate model, and Analytical verification will take the form of calculating the analytic general solutions (Section 2.1). We plan to combine these results to highlight how our theoretical models compare to experimental slow-motion footage.

3.2. *Extension*

We will first extend our model by considering the effect of friction and altering our original model to account for this force. Essentially we will be trying to find the coefficients of drag. We suspect that that these coefficients will grow with increasing size and speed of oscillations of the chain or beads. We will use our experimental verification results to aid in this process.

Additionally, we will repeat the experimental steps outlined above with the chain and beads submerged in water. We are interested in comparing the behavior of these materials under water with our model. We suspect that results may be similar to that of the behavior described when air friction is considered.

3.3. *Exploration*

If time permits, we will also consider our the shape of the rotating chain immersed in water. We are motivated by our extension (Section 3.2), realizing that the chain is inevitably immersed in a fluid (air), but consider exploring if dynamics of the rotating chain in a more viscous fluid (water). We are further motivated by the design of vertical axial wind turbines (VAWTS). One of the most famous VAWT is the Darrieus turbine (Figure 7), which uses airfoils to create lift that maintains rotation. The Darrieus turbine has the shape of a troposkein which is the curve an idealized rope (in the absense of gravity) assumes when anchored at its ends and spun around its long axis at a constant angular velocity. The troposkien shape reduces the flatwise bending stresses due to centrifugal and gravitational forces as the blade tends to displace less from its original shape. A 1986 Sandia report discusses an improved model of the Darrieus turbine considering the shape a rope assumes under both rotation and the effects of gravity [Ashwill (1986)]. Darrieus turbine has also been studied extensively using CFD and experimental methods to optimize the performance [Wenlong *et al.* (2013)]. There has also been research interest considering catenaries in viscous fluid, [Chakrabarti & Hanna (2016)]. Therefore, we are motivated both by our extension section (3.2) and research interest on the shape of chains and ropes immersed in viscous fluid for both intrinsic value and possible engineering application.

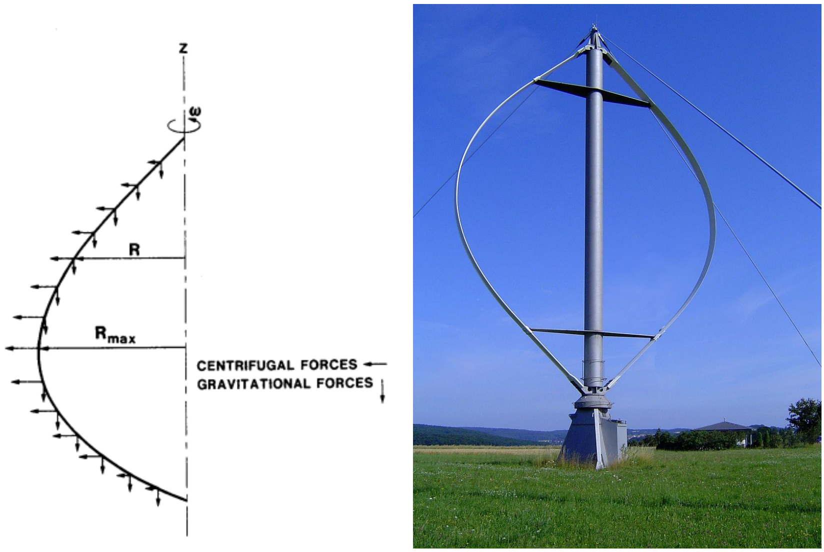

Figure 7: Sandia Report Improved Turbine Left, Darrieus Turbine Right
[Ashwill (1986)][Wacker (2005)]

4. Other Applications

There are an immense number of applications and extensions to apply to our basic model, and we list some here to perhaps motivate future projects. One of these is the analysis of more dramatic oscillations than our model permitted. Some researchers have come across the self-knot phenomena. In the presence of large enough oscillations, the chain will overlap itself and essentially form a knot with itself [Belmonte (2001)].

Another interesting case to consider is that of a magnetic chain. Similar to our inverted, buoyant material floating chain, the magnetic chain has links which attraction to each other hold the chain upright when fixed at its lowest point [Schönke & Fried (2017)]. Other interesting applications Petit & Rouchon (2001)Wang & Wang (2010). These include using a non-uniform density chain or material, using a driven system (eg. water hose), sliding the chain in addition to rotating it, and applying control theory stabilization methods.

Using a buoyant material and attaching it at the bottom of our water-filled tank is also a variation of our original build that could provide some interesting results. However, it will be a challenge rigging a system in which we can use a drill to rotate the material in the tank from below.

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