In class, we talked about

$$\int_0^1 \frac{1}{\sqrt{x^3+1}} dx.$$

I made a common error in these problems, which is to not look carefully at the integrand. Notice that the integrand never becomes infinite, so there is no need to look at convergence at all. On the other hand,

$$\int_0^1 \frac{1}{\sqrt{x^3-1}} dx$$

is not well defined, since on $[0,1]$, $x^3 - 1 < 0$ and so we cannot find a square root. Thus neither of my examples were very good.

Here is a better one. Consider

$$\int_0^1 \frac{1}{\sqrt[3]{1-x^2}} dx$$

The integrand goes to infinity at $x = 1$ and so it is improper. Now, we may try to compare it to another integral whose integrand goes to infinity, such as

$$\int_0^1 \frac{1}{(1-x)^{2/3}} dx.$$

Notice that this integral converges and is equal to

$$\begin{aligned}\int_0^1 \frac{1}{(1-x)^{2/3}} dx &= -\int_1^0 \frac{1}{w^{2/3}} dw \\ &= \lim_{A \to 0+} \left(-3\, w^{1/3} \Big|_1^A \right) \\ &= 3.\end{aligned}$$

Now, we may try to compare. We can try to work backwards and try to get a comparison with $\frac{1}{\sqrt[3]{2(1-x)^2}}$ as follows. We want

$$\begin{aligned}\frac{1}{\sqrt[3]{1-x^2}} &\le \frac{1}{\sqrt[3]{(1-x)^2}} \\ \sqrt[3]{(1-x)^2} &\le \sqrt[3]{1-x^2} \\ (1-x)^2 &\le 1-x^2 \\ 1-2x+x^2 &\le 1-x^2 \\ x-2 &\le -x \\ x &\le 1\end{aligned}$$

So, working backwards we find that if $x \leq 1$ then $\frac{1}{\sqrt[3]{1-x^2}} \leq \frac{1}{\sqrt[3]{(1-x)^2}}$. Since

$$\int_0^1 \frac{1}{\sqrt[3]{(1-x)^2}} dx$$

converges, then

$$\int_0^1 \frac{1}{\sqrt[3]{1-x^2}} dx$$

must converge.