Solutions of Practice for Test 3

1) Chapter 10 Check Your Understanding: 1, 3, 5, 7
 Answers in the back of the book
2) Chapter 10 Review: 9, 11, 15, 28, 33(a,b)
 9, 11, 15, and 33 are in the back of the book. For 28,

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \cdots \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \\
1 - \cos x &= -\frac{x^2}{2} + \frac{x^4}{4!} \cdots \\
e^x - 1 &= x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \\
\arctan x &= x - \frac{x^3}{3} + \cdots \\
x\sqrt{1-x} &= x - \frac{x^2}{2} - \frac{x^3}{8} + \cdots
\end{aligned}$$

for x close to zero, so for small positive x,

$$1 - \cos x \le x\sqrt{1-x} \le \ln(1+x) \le \arctan x \le \sin x \le x \le e^x - 1$$

3) Chapter 11 Check Your Understanding: 1, 3, 5, 9, 11
 These are in the back of the book.
4) Chapter 11 Review: 3, 7, 19, 29, 31, 33
 These are in the back of the book.

5) State if the following are true or false. Be sure to justify your answers and correct if false.

a) If the power series $\sum_{n=0}^{\infty} b_n (x-4)^n$ has radius of convergence 5 then the series converges for $-5 \le x \le 5$.

False: it converges for $|x-4| < 5$ or $-1 < x < 9$

b) The Taylor series for $\frac{1}{1-x}$ centered at $x = 0$ has radius of convergence equal to infinity.

False: It's radius of convergence is 1.

c) The differential equation

$$\frac{dx}{dt} = \sin(x) + t$$

is separable (i.e. it can be solved using separation of variables).

False: It is not separable (it would be if, for instance, one of the two functions in the sum were not present).

d) The differential equation

$$\frac{dP}{dt} = -P$$

has a stable equilibrium solution.

True. It has a stable equilibrium solution $P(t) = 0$.

e) To solve the differential equation $\frac{dy}{dt} = t^3 + y$ with $y = 2$ when $t = 0$, if one uses Euler's method with $\Delta t = 0.3$, he/she will get the approximations $y(0.3) = 1$ and $y(0.6) = 4$.

False: Euler's method yields $x_0 = 0$, $y_0 = 2$, $x_1 = 0.3$, $y_1 = y_0 + \left[x_0^3 + y_0\right] 0.3 = 2 + (2)(0.3) = 2.6$, $x_2 = 0.6$, and $y_2 = 2.6 + \left((0.3)^3 + 2.6\right)(0.3) = 3.3881$. Hence the approximation is $y(0.3) = 2.6$ and $y(0.6) = 3.3881$.

6) Find the first 4 nonzero terms in the Taylor series for the following functions.

a) $\sin(x)$ at $x = 0$.

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

b) $\cos 3x$ at $x = 0$.

$$1 - \frac{9x^2}{2} + \frac{3^4 x^4}{4!} - \frac{3^6 x^6}{6!}$$

c) xe^x at $x = 0$.

$$x + x^2 + \frac{x^3}{2} + \frac{x^4}{3!}$$

d) $\frac{1}{x}$ at $x = 1$.

$$1 - (x-1)^1 + (x-1)^2 - (x-1)^3$$

e) $\arctan(x)$ at $x = 0$.

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

f) $\ln(-x)$ at $x = -1$.

$$-(x+1)^1 - \frac{1}{2}(x+1)^2 - \frac{1}{3}(x+1)^3 - \frac{1}{4}(x+1)^4$$

g) $x^4 + 2x$ at $x = 0$.

$$2x + x^4$$

h) $x^4 + 2x$ at $x = 2$.

$$20 + 34(x-2)^1 + 24(x-2)^2 + 8(x-2)^3 + (x-2)^4$$

f) $\frac{1}{(1-x)^3}$ at $x = 0$.

$$1 + 3x + 6x^2 + 10x^3$$

7) Find the following by recognizing the appropriate Taylor series.
a) $1+(0.1)^2+(0.1)^3+(0.1)^4+(0.1)^5+\cdots = \frac{1}{1-0.1}-0.1 = \frac{10}{9}-\frac{1}{10}=\frac{91}{90}$
b) $\sum_{n=1}^{\infty}\frac{(0.3)^n}{n} = -\ln(1-0.3) = -\ln(0.7)$
c) $1-\frac{1}{2}+\frac{1}{6}-\frac{1}{24}+\frac{1}{120}-\cdots = 1-e^{-1}$
d) $1+x^2+x^4+x^6+x^8+\cdots = \frac{1}{1-x^2}$
e) $\sum_{n=1}^{\infty} 2^n\frac{x^n}{n!} = e^{2x}-1$
f) $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+3}}{(2n+3)!} = x-\sin x$

8) Compute the radius of convergence for the following power series.
a) $\sum_{n=1}^{\infty}\frac{n^2}{(n+1)^2}(x-3)^n$
Radius of convergence = 1.
b) $\sum_{n=0}^{\infty}\frac{9^n(n+1)^3}{n!}(x+1)^n$
Radius of convergence = ∞.
c) $\sum_{n=3}^{\infty} n(-3)^n x^{2n+1}$
Radius of convergence = $1/\sqrt{3}$.
d) $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^3}(x-4)^{2n}$

Radius of convergence = 1.

9) a) Use Taylor series to put the following functions in order from smallest to largest for positive values of x near zero: $1-x+x^2$, $\frac{1}{1+x}$, e^{-x}, $\cos x$, $1-x$
Note that

$$\frac{1}{1+x} = 1-x+x^2-x^3+\cdots$$
$$e^{-x} = 1-x+\frac{x^2}{2}-\frac{x^3}{3!}+\cdots$$
$$\cos x = 1-\frac{x^2}{2}+\frac{x^4}{4!}$$

so for small positive values of x,

$$1-x \le e^{-x} \le \frac{1}{1+x} \le 1-x+x^2 \le \cos x$$

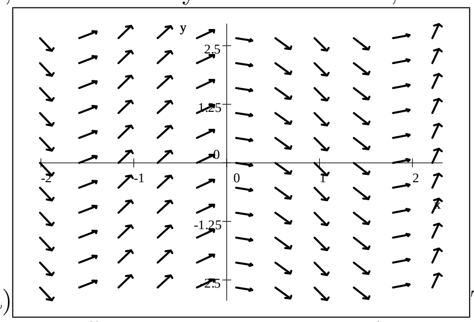
b) Now dow the same for negative values of x near zero.
For negative values,

$$\cos x \le 1-x \le e^{-x} \le \frac{1}{1+x} \le 1-x+x^2$$

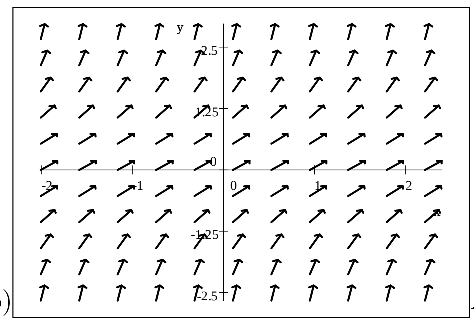
10) Give reasons why each of the following slope fields cannot be the slope field for the differential equation

$$\frac{dy}{dx} = (x-1)\,y.$$

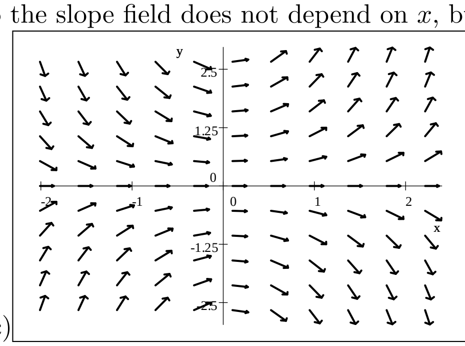
Note, there are many reasons for each, so I only give a couple possibilities.

a) 

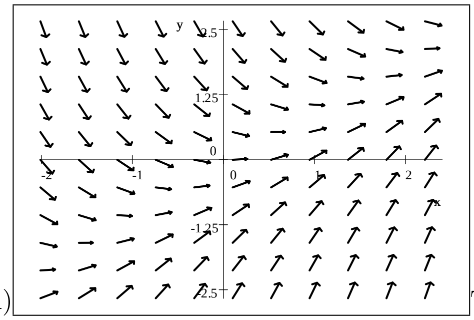
The slope field does not depend on y but the differential equation does. Also the differential equation indicates that all vectors when $x > 1$ and $y > 0$ should have positive slope, which is clearly false. Finally, all slopes when $y = 0$ should be zero according to the differential equation, but it is not true on the slope field.

b) 

All slopes when $y = 0$ should be zero according to the differential equation, but it is not true on the slope field. Also the slope field does not depend on x, but the differential equation does.

c) 

The slopes should be equal to zero if $x = 1$ but this is not the case.

d) 

The slopes should be equal to zero if $x = 1$ or $y = 0$ but this is not the case.

$\sum_{n=1}^{\infty} \frac{1}{2^n}$, $\sum_{n=1}^{\infty} \frac{1}{2^2}$, $\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$, $\sum_{n=1}^{\infty} n^{-1}$, $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$, $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

11) Decide which of the following series converge and which diverge. Justify your answer:

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges because it is a geometric series with ratio 1/2..

$\sum_{n=1}^{\infty} \frac{1}{2^2}$ diverges since it is a multiple of $1 + 1 + 1 + 1 + \cdots$

$\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$ converges by integral test since $\int_1^{\infty} \frac{1}{x^{1.1}} dx$ converges (or p-series with $p < 1$).

$\sum_{n=1}^{\infty} n^{-1}$ diverges (it is the harmonic series, so there is the argument we gave in class as well as the integral test since $\int_1^{\infty} \frac{1}{x} dx$ diverges).

$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges because $\sum \frac{1}{n^2}$ converges and $\lim_{n\to\infty} \frac{1}{n^2-1} / \frac{1}{n^2} = 1$ so the limit comparison test gives the result..

$\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges. Use limit comparison with $\sum \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges by integral test, since $\int_1^{\infty} \frac{1}{x \ln x} dx = \int_0^{\infty} \frac{1}{u} du$ which diverges.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test.

12)

Answers in the back of the book

13)

Answers in the back of the book