Chapter 12: Partial Differential Equations

1. Partial differential equations

- A partial differential equation (PDE) is an equation giving a relation between a function of two or more variables, u, and its partial derivatives.
- The order of the PDE is the order of the highest partial derivative of u that appears in the PDE.
- A PDE is linear if it is linear in u and in its partial derivatives. A linear PDE is homogeneous if all of its terms involve either u or one of its partial derivatives.
- A solution to a PDE is a function u that satisfies the PDE.
- Finding a specific solution to a PDE typically requires an initial condition as well as boundary conditions.

Examples

- Check that $u = f(x+ct) + g(x-ct)$, where f and g are two smooth functions, is a solution (called d'Alembert's solution) to the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

- Is the two-dimensional wave equation (given below) linear?

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

- What is the order of the heat equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$?
- The Laplace equation reads $\Delta u = 0$, where Δ is the two- or three-dimensional Laplacian. Is this equation homogeneous?

2. The one-dimensional wave equation

- The one-dimensional wave equation models the 2-dimensional dynamics of a vibrating string which is stretched and clamped at its end points (say at $x = 0$ and $x = L$).
- The function $u(x, t)$ measures the deflection of the string and satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 \propto T, \;\; T \equiv \text{tension of the string}$$

with Dirichlet boundary conditions

$$u(0, t) = u(L, t) = 0, \qquad \text{for all } t \geq 0.$$

- In what follows, we assume that the initial conditions are

$$u(x, 0) = f(x), \qquad u_t(x, 0) \equiv \frac{\partial u}{\partial t}(x, 0) = g(x), \qquad \text{for } x \in [0,\ L].$$

Solution by separation of variables

- We look for a solution $u(x,t)$ in the form $u(x,t) = F(x)G(t)$.
- Substitution into the one-dimensional wave equation gives

$$\frac{1}{c^2\,G(t)}\,\frac{d^2G}{dt^2} = \frac{1}{F}\,\frac{d^2F}{dx^2}.$$

  Since the left-hand side is a function of t only and the right-hand side is a function of x only, and since x and t are independent, the two terms must be equal to some constant k.

- Imposing the boundary conditions gives solutions of the form

$$u_n(x,t) = \left[a_n \cos\left(c\,n\,\frac{\pi t}{L}\right) + b_n \sin\left(c\,n\,\frac{\pi t}{L}\right)\right] \sin\left(n\,\frac{\pi x}{L}\right),$$

  for $n = 1,\ 2,\ \cdots$, where $k = -\left(\frac{n\pi}{L}\right)^2$, and the a_n's and b_n's are arbitrary constants.

Solution by separation of variables (continued)

- The functions $u_n(x,t)$ are called the normal modes of the vibrating string. The n-th normal mode has $n-1$ nodes, which are points in space where the string does not vibrate.

- The general solution to the one-dimensional wave equation with Dirichlet boundary conditions is therefore a linear combination of the normal modes of the vibrating string,

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} C_n u_n(x,t) \\ &= \sum_{n=1}^{\infty} \left[A_n \cos\left(c\, n \frac{\pi t}{L}\right) + B_n \sin\left(c\, n \frac{\pi t}{L}\right)\right] \sin\left(n \frac{\pi x}{L}\right), \end{aligned}$$

where $A_n = C_n a_n$ and $B_n = C_n b_n$.

Solution by separation of variables (continued)

- The coefficients of the above expansion are found by imposing the initial conditions.
- Since $u_n(x,0)$ and $\dfrac{\partial u_n}{\partial t}(x,0)$ are proportional to $\sin(n\pi x/L)$, imposing the initial conditions amounts to finding the orthogonal expansions of the functions $f(x)$ and $g(x)$ on $\{\sin(n\pi x/L),\ n = 1, 2, \cdots\}$.
- Therefore, with $U_n(x) = \sin\left(n\dfrac{\pi x}{L}\right)$,

$$A_n = \frac{\langle u(x,0), U_n(x)\rangle}{\|U_n\|^2} = \frac{2}{L}\int_0^L f(x)\ \sin\left(n\frac{\pi x}{L}\right)\ dx,$$

$$B_n = \frac{L}{c\,n\,\pi}\frac{\langle u_t(x,0), U_n(x)\rangle}{\|U_n(x)\|^2} = \frac{2}{L}\int_0^L \frac{L}{c\,n\,\pi}\ g(x)\sin\left(n\frac{\pi x}{L}\right)\ dx.$$

Solution by separation of variables (continued)

- **Example:** Show that the solution to $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with Dirichlet boundary conditions on $[0,\ 1]$ and initial condition

$$u(x,0) = \begin{cases} \frac{x}{5} & \text{if } 0 \le x \le 0.5 \\ \frac{1-x}{5} & \text{if } 0.5 \le x \le 1 \end{cases}, \qquad \frac{\partial u}{\partial t}(x,0) = 0,$$

is of the form

$$u(x,t) = \frac{4}{5\pi^2}\Big[\sin(\pi x)\cos(c\pi t) - \frac{1}{9}\sin(3\pi x)\cos(3c\pi t) + \frac{1}{25}\sin(5\pi x)\cos(5c\pi t) + \cdots\Big].$$

- Experiment with the *Vibrating String* MATLAB GUI.

3. The two-dimensional wave equation

- The two-dimensional wave equation models the 3-dimensional dynamics of a stretched elastic membrane clamped at its boundary.
- The function $u(x, y, t)$ measures the vertical displacement of the membrane (think of a drum for instance) and satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = c^2 \nabla^2 u,$$

  where c^2 is proportional to the tension of the membrane.
- The boundary conditions (Dirichlet) are $u = 0$ on the boundary of the membrane and the initial conditions are of the form

$$u(x, y, 0) = f(x, y), \qquad u_t(x, y, 0) \equiv \frac{\partial u}{\partial t}(x, y, 0) = g(x, y).$$

Rectangular membrane

- For a rectangular membrane, we use separation of variables in cartesian coordinates, i.e. we let

$$u(x,y,t) = F(x,y)G(t),$$

where the functions F, and G are to be determined.

- Substitution into the wave equation leads to

$$\frac{1}{c^2 G}\frac{d^2 G}{dt^2} = \frac{1}{F}\nabla^2 F = -\nu^2,$$

where ν is a real constant.

- The function F therefore satisfies Helmholtz's equation, $\nabla^2 F + \nu^2 F = 0$, which can also be solved by separation of variables, i.e. by letting $F(x,y) = H_1(x)H_2(y)$.

Rectangular membrane (continued)

- As before, imposing the boundary conditions leads to a collection of normal modes for the square membrane, which are

$$u_{mn}(x, y, t) = [a_{mn}\cos(\lambda_{mn}t) + b_{mn}\sin(\lambda_{mn}t)] \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),$$

where

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

and the membrane is the rectangle $0 \le x \le a$, $0 \le y \le b$.

- The next step is to impose the initial conditions.

Rectangular membrane (continued)

- Since the wave equation is linear, the solution u can be written as a linear combination (i.e. a superposition) of the normal modes for the given boundary conditions. In other words, we write

$$\begin{aligned} u(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\, u_{mn}(x,y,t) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Big([A_{mn}\cos(\lambda_{mn}t) + B_{mn}\sin(\lambda_{mn}t)] \\ &\qquad \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\Big)\,, \end{aligned}$$

where $A_{mn} = C_{mn}\, a_{mn}$ and $B_{mn} = C_{mn}\, b_{mn}$.

Rectangular membrane (continued)

- The coefficients A_{mn} and B_{mn} are found by writing the orthogonal expansions of the initial conditions $f(x, y)$ and $g(x, y)$ as double Fourier sine series. The corresponding dot product is defined by

$$\langle f, g \rangle = \int_0^a \int_0^b f(x, y)\, g(x, y)\; dy\; dx.$$

- The presence of nodal lines in the normal modes may lead to the existence of nodal curves in the solution $u(x, y, t)$.
- Experiment with the *Rectangular Elastic Membrane* MATLAB GUI.

Circular membrane

- For a circular membrane, it is more appropriate to write the Laplacian in polar coordinates, so that $u = u(r, \theta, t)$ solves

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right).$$

- If the membrane has radius R, the boundary conditions are

$$u(R, \theta, t) = 0, \qquad \text{for all } t.$$

- For radially symmetric solutions (i.e. if $u_\theta(r, \theta, t) = 0$), the method of separation of variables leads to normal modes in terms of Bessel functions. Finding a specific solution amounts to finding an orthogonal expansion of the initial conditions, this time in terms of Fourier-Bessel series.
- Experiment with the *Circular Elastic Membrane* MATLAB GUI.

4. The one-dimensional heat equation on a finite interval

- The one-dimensional heat equation models the diffusion of heat (or of any diffusing quantity) through a homogeneous one-dimensional material (think for instance of a rod).
- The function $u(x, t)$ measures the temperature of the rod at point x and at time t. It satisfies the heat equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 \equiv \text{ diffusion coefficient.}$$

- Typical boundary conditions are of one of the following types,
  - Dirichlet: $u(0, t) = u(L, t) = 0$ for all $t \geq 0$;
  - Neumann: $\dfrac{\partial u}{\partial x}(0, t) = \dfrac{\partial u}{\partial x}(L, t) = C$ for all $t \geq 0$, where C is a given constant (often, $C = 0$);

  where we assume that the end points of the rod are at $x = 0$ and $x = L$.

The one-dimensional heat equation (continued)

- One can also consider mixed boundary conditions, for instance Dirichlet at $x = 0$ and Neumann at $x = L$.
- The initial condition is given in the form

$$u(x, 0) = f(x),$$

where f is a known function.

- In this section, we solve the heat equation with Dirichlet boundary conditions. As for the wave equation, we use the method of separation of variables.
- Setting $u(x, t) = F(x)G(t)$ gives

$$\frac{1}{c^2 G}\frac{dG}{dt} = \frac{1}{F}\frac{d^2F}{dx^2} = k,$$

where k is some constant to be determined.

Separation of variables

- As for the wave equation, the boundary conditions can only be satisfied if we impose $k < 0$, say $k = -\nu^2$.

- The solution to $\dfrac{d^2F}{dx^2} = k\,F = -\nu^2 F$ is then

$$F(x) = b_n \sin\left(n\frac{\pi x}{L}\right), \qquad n = 1, 2, \cdots,$$

where ν has to satisfy $\nu = n\pi/L$.

- After solving for $G(t)$, we obtain an infinite number of modes,

$$u_n(x,t) = b_n \sin\left(n\frac{\pi x}{L}\right)\, \exp\left[-\left(\frac{c\,n\,\pi}{L}\right)^2 t\right].$$

where $n = 1, 2, \cdots$.

Separation of variables (continued)

- Since the heat equation is linear, its general solution in the presence of Dirichlet boundary conditions is given by a linear combination (or superposition) of the modes u_n, i.e.

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} C_n u_n(x,t) \\ &= \sum_{n=1}^{\infty} B_n \sin\left(n\frac{\pi x}{L}\right) \exp\left[-\left(\frac{c\,n\,\pi}{L}\right)^2 t\right], \end{aligned}$$

where $B_n = C_n b_n$.

- The initial condition reads $f(x) = \sum_{n=1}^{\infty} B_n \sin\left(n\frac{\pi x}{L}\right)$, and the coefficients B_n can therefore be obtained by finding the half-range sine expansion of $f(x)$.

Separation of variables (continued)

- In other words, we have

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(n \frac{\pi x}{L}\right)\, dx.$$

- For an insulated rod (i.e. for Neumann boundary conditions with $C = 0$), the solution is of the form

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(n \frac{\pi x}{L}\right)\, \exp\left[-\left(\frac{c\, n\, \pi}{L}\right)^2 t\right],$$

  and the A_n are found by writing the half-range cosine expansion of the initial condition $f(x)$.

- Experiment with the *One-dimensional Heat Equation* MATLAB GUI.

5. The one-dimensional heat equation on the whole line

- To solve the one-dimensional heat equation on the whole line, one first formally takes the Fourier transform of the heat equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \Longrightarrow \frac{d\widehat{u}_k}{dt} = -c^2 k^2\, \widehat{u}_k.$$

- The initial condition, $u(x,0) = f(x)$ reads $\widehat{u}_k(0) = \widehat{f}(k)$, and the solution is therefore

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k) \exp(-c^2 k^2 t) \exp(i\,k\,x)\; dk.$$

- We can recognize this integral as the inverse Fourier transform of a product of two Fourier transforms, $\widehat{f}(k)$ and $\widehat{g}(k)$, where $\widehat{g}(k) = \exp(-c^2 k^2 t)$.

Method of convolution

- Since we know that $g(x) = \frac{1}{\sqrt{2c^2t}} \exp\left(-\frac{x^2}{4c^2t}\right)$, and since the inverse Fourier transform of a product is the convolution of the inverse transforms times $\frac{1}{\sqrt{2\pi}}$, we therefore have

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(y) \exp\left(-\frac{(x-y)^2}{4c^2t}\right) \, dy.$$

- **Example:** Solve the heat equation on the whole line with initial condition $u(x,0) = 1$ if $|x| < 1$ and $u(x,0) = 0$ otherwise.

- Experiment with the *Heat Equation on the Whole Line* MATLAB GUI.