

## Chapter 12: Partial Differential Equations

## 1. Partial differential equations

- A **partial differential equation** (PDE) is an equation giving a relation between a function of two or more variables,  $u$ , and its partial derivatives.
- The **order** of the PDE is the order of the highest partial derivative of  $u$  that appears in the PDE.
- A PDE is **linear** if it is linear in  $u$  and in its partial derivatives. A linear PDE is **homogeneous** if all of its terms involve either  $u$  or one of its partial derivatives.
- A **solution** to a PDE is a function  $u$  that satisfies the PDE.
- Finding a specific solution to a PDE typically requires an **initial condition** as well as **boundary conditions**.

## Examples

- Check that  $u = f(x + ct) + g(x - ct)$ , where  $f$  and  $g$  are two smooth functions, is a solution (called **d'Alembert's solution**) to the **one-dimensional wave equation**,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

- Is the **two-dimensional wave equation** (given below) linear?

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

- What is the order of the **heat equation**  $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ ?
- The **Laplace equation** reads  $\Delta u = 0$ , where  $\Delta$  is the two- or three-dimensional Laplacian. Is this equation homogeneous?

## 2. The one-dimensional wave equation

- The one-dimensional wave equation models the 2-dimensional dynamics of a **vibrating string** which is stretched and clamped at its end points (say at  $x = 0$  and  $x = L$ ).
- The function  $u(x, t)$  measures the deflection of the string and satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 \propto T, \quad T \equiv \text{tension of the string}$$

with **Dirichlet boundary conditions**

$$u(0, t) = u(L, t) = 0, \quad \text{for all } t \geq 0.$$

- In what follows, we assume that the **initial conditions** are
- $$u(x, 0) = f(x), \quad u_t(x, 0) \equiv \frac{\partial u}{\partial t}(x, 0) = g(x), \quad \text{for } x \in [0, L].$$

## Solution by separation of variables

- We look for a solution  $u(x, t)$  in the form  $u(x, t) = F(x)G(t)$ .
- Substitution** into the one-dimensional wave equation gives

$$\frac{1}{c^2 G(t)} \frac{d^2 G}{dt^2} = \frac{1}{F} \frac{d^2 F}{dx^2}.$$

Since the left-hand side is a function of  $t$  only and the right-hand side is a function of  $x$  only, and since  $x$  and  $t$  are independent, **the two terms must be equal to some constant  $k$** .

- Imposing the **boundary conditions** gives solutions of the form

$$u_n(x, t) = \left[ a_n \cos\left(c n \frac{\pi t}{L}\right) + b_n \sin\left(c n \frac{\pi t}{L}\right) \right] \sin\left(n \frac{\pi x}{L}\right),$$

for  $n = 1, 2, \dots$ , where  $k = -\left(\frac{n\pi}{L}\right)^2$ , and the  $a_n$ 's and  $b_n$ 's are arbitrary constants.

## Solution by separation of variables (continued)

- The functions  $u_n(x, t)$  are called the **normal modes** of the vibrating string. The  $n$ -th normal mode has  $n - 1$  **nodes**, which are points in space where the string does not vibrate.
- The general solution to the one-dimensional wave equation with Dirichlet boundary conditions is therefore **a linear combination of the normal modes** of the vibrating string,

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} C_n u_n(x, t) \\ &= \sum_{n=1}^{\infty} \left[ A_n \cos\left(c n \frac{\pi t}{L}\right) + B_n \sin\left(c n \frac{\pi t}{L}\right) \right] \sin\left(n \frac{\pi x}{L}\right), \end{aligned}$$

where  $A_n = C_n a_n$  and  $B_n = C_n b_n$ .

## Solution by separation of variables (continued)

- The **coefficients** of the above expansion are **found by imposing the initial conditions**.
- Since  $u_n(x, 0)$  and  $\frac{\partial u_n}{\partial t}(x, 0)$  are proportional to  $\sin(n\pi x/L)$ , imposing the initial conditions amounts to finding the **orthogonal expansions** of the functions  $f(x)$  and  $g(x)$  on  $\{\sin(n\pi x/L), n = 1, 2, \dots\}$ .
- Therefore, with  $U_n(x) = \sin\left(n \frac{\pi x}{L}\right)$ ,

$$A_n = \frac{\langle u(x, 0), U_n(x) \rangle}{\|U_n\|^2} = \frac{2}{L} \int_0^L f(x) \sin\left(n \frac{\pi x}{L}\right) dx,$$

$$B_n = \frac{L}{c n \pi} \frac{\langle u_t(x, 0), U_n(x) \rangle}{\|U_n(x)\|^2} = \frac{2}{L} \int_0^L \frac{L}{c n \pi} g(x) \sin\left(n \frac{\pi x}{L}\right) dx.$$

## Solution by separation of variables (continued)

- Example:** Show that the solution to  $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$  with Dirichlet boundary conditions on  $[0, 1]$  and initial condition

$$u(x, 0) = \begin{cases} \frac{x}{5} & \text{if } 0 \leq x \leq 0.5 \\ \frac{1-x}{5} & \text{if } 0.5 \leq x \leq 1 \end{cases}, \quad \frac{\partial u}{\partial t}(x, 0) = 0,$$

is of the form

$$u(x, t) = \frac{4}{5\pi^2} \left[ \sin(\pi x) \cos(c\pi t) - \frac{1}{9} \sin(3\pi x) \cos(3c\pi t) + \frac{1}{25} \sin(5\pi x) \cos(5c\pi t) + \dots \right].$$

- Experiment with the Vibrating String MATLAB GUI.

### 3. The two-dimensional wave equation

- The **two-dimensional wave equation** models the 3-dimensional dynamics of a stretched **elastic membrane** clamped at its boundary.
- The function  $u(x, y, t)$  measures the vertical displacement of the membrane (think of a drum for instance) and satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = c^2 \nabla^2 u,$$

where  $c^2$  is proportional to the tension of the membrane.

- The **boundary conditions** (Dirichlet) are  $u = 0$  on the **boundary of the membrane** and the **initial conditions** are of the form

$$u(x, y, 0) = f(x, y), \quad u_t(x, y, 0) \equiv \frac{\partial u}{\partial t}(x, y, 0) = g(x, y).$$

### Rectangular membrane

- For a **rectangular membrane**, we use **separation of variables** in cartesian coordinates, i.e. we let

$$u(x, y, t) = F(x, y)G(t),$$

where the functions  $F$ , and  $G$  are to be determined.

- **Substitution** into the wave equation leads to

$$\frac{1}{c^2 G} \frac{d^2 G}{dt^2} = \frac{1}{F} \nabla^2 F = -\nu^2,$$

where  $\nu$  is a real constant.

- The function  $F$  therefore satisfies **Helmholtz's equation**,  $\nabla^2 F + \nu^2 F = 0$ , which can also be solved by separation of variables, i.e. by letting  $F(x, y) = H_1(x)H_2(y)$ .

### Rectangular membrane (continued)

- As before, imposing the **boundary conditions** leads to a collection of **normal modes** for the square membrane, which are

$$u_{mn}(x, y, t) = [a_{mn} \cos(\lambda_{mn} t) + b_{mn} \sin(\lambda_{mn} t)] \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right),$$

where

$$\lambda_{mn} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

and the membrane is the rectangle  $0 \leq x \leq a$ ,  $0 \leq y \leq b$ .

- The next step is to **impose the initial conditions**.

### Rectangular membrane (continued)

- Since the wave equation is **linear**, the solution  $u$  can be written as a linear combination (i.e. a **superposition**) of the normal modes for the given boundary conditions. In other words, we write

$$\begin{aligned} u(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} u_{mn}(x, y, t) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( [A_{mn} \cos(\lambda_{mn} t) + B_{mn} \sin(\lambda_{mn} t)] \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \right), \end{aligned}$$

where  $A_{mn} = C_{mn} a_{mn}$  and  $B_{mn} = C_{mn} b_{mn}$ .

## Rectangular membrane (continued)

- The coefficients  $A_{mn}$  and  $B_{mn}$  are found by writing the **orthogonal expansions** of the initial conditions  $f(x, y)$  and  $g(x, y)$  as **double Fourier sine series**. The corresponding **dot product** is defined by

$$\langle f, g \rangle = \int_0^a \int_0^b f(x, y) g(x, y) dy dx.$$

- The presence of **nodal lines** in the normal modes may lead to the existence of **nodal curves** in the solution  $u(x, y, t)$ .
- Experiment with the Rectangular Elastic Membrane MATLAB GUI.

## Circular membrane

- For a **circular membrane**, it is more appropriate to write the **Laplacian in polar coordinates**, so that  $u = u(r, \theta, t)$  solves

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right).$$

- If the membrane has radius  $R$ , the **boundary conditions** are  $u(R, \theta, t) = 0$ , for all  $t$ .
- For **radially symmetric solutions** (i.e. if  $u_\theta(r, \theta, t) = 0$ ), the method of **separation of variables** leads to **normal modes** in terms of Bessel functions. Finding a specific solution amounts to finding an **orthogonal expansion** of the initial conditions, this time in terms of **Fourier-Bessel series**.
- Experiment with the Circular Elastic Membrane MATLAB GUI.

## 4. The one-dimensional heat equation on a finite interval

- The **one-dimensional heat equation** models the diffusion of heat (or of any diffusing quantity) through a homogeneous one-dimensional material (think for instance of a rod).
- The function  $u(x, t)$  measures the temperature of the rod at point  $x$  and at time  $t$ . It satisfies the heat equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 \equiv \text{diffusion coefficient.}$$

- Typical **boundary conditions** are of one of the following types,
  - Dirichlet:**  $u(0, t) = u(L, t) = 0$  for all  $t \geq 0$ ;
  - Neumann:**  $\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = C$  for all  $t \geq 0$ , where  $C$  is a given constant (often,  $C = 0$ );

where we assume that the end points of the rod are at  $x = 0$  and  $x = L$ .

## The one-dimensional heat equation (continued)

- One can also consider **mixed boundary conditions**, for instance Dirichlet at  $x = 0$  and Neumann at  $x = L$ .
- The **initial condition** is given in the form

$$u(x, 0) = f(x),$$

where  $f$  is a known function.

- In this section, we solve the heat equation with **Dirichlet boundary conditions**. As for the wave equation, we use the method of **separation of variables**.
- Setting  $u(x, t) = F(x)G(t)$  gives

$$\frac{1}{c^2 G} \frac{dG}{dt} = \frac{1}{F} \frac{d^2 F}{dx^2} = k,$$

where  $k$  is some constant to be determined.

## Separation of variables

- As for the wave equation, the **boundary conditions** can only be satisfied if we impose  $k < 0$ , say  $k = -\nu^2$ .

- The solution to  $\frac{d^2 F}{dx^2} = k F = -\nu^2 F$  is then

$$F(x) = b_n \sin\left(n\frac{\pi x}{L}\right), \quad n = 1, 2, \dots,$$

where  $\nu$  has to satisfy  $\nu = n\pi/L$ .

- After solving for  $G(t)$ , we obtain **an infinite number of modes**,

$$u_n(x, t) = b_n \sin\left(n\frac{\pi x}{L}\right) \exp\left[-\left(\frac{c n \pi}{L}\right)^2 t\right].$$

where  $n = 1, 2, \dots$ .

## Separation of variables (continued)

- Since the heat equation is **linear**, its general solution in the presence of Dirichlet boundary conditions is given by a **linear combination** (or superposition) of the modes  $u_n$ , i.e.

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} C_n u_n(x, t) \\ &= \sum_{n=1}^{\infty} B_n \sin\left(n\frac{\pi x}{L}\right) \exp\left[-\left(\frac{c n \pi}{L}\right)^2 t\right], \end{aligned}$$

where  $B_n = C_n b_n$ .

- The **initial condition** reads  $f(x) = \sum_{n=1}^{\infty} B_n \sin\left(n\frac{\pi x}{L}\right)$ , and the coefficients  $B_n$  can therefore be obtained by finding the **half-range sine expansion** of  $f(x)$ .

## Separation of variables (continued)

- In other words, we have

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(n\frac{\pi x}{L}\right) dx.$$

- For an insulated rod (i.e. for **Neumann boundary conditions** with  $C = 0$ ), the solution is of the form

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos\left(n\frac{\pi x}{L}\right) \exp\left[-\left(\frac{c n \pi}{L}\right)^2 t\right],$$

and the  $A_n$  are found by writing the **half-range cosine expansion** of the initial condition  $f(x)$ .

- Experiment with the One-dimensional Heat Equation MATLAB GUI.

## 5. The one-dimensional heat equation on the whole line

- To solve the one-dimensional heat equation on the whole line, one first formally **takes the Fourier transform of the heat equation**,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \implies \frac{d\hat{u}_k}{dt} = -c^2 k^2 \hat{u}_k.$$

- The initial condition,  $u(x, 0) = f(x)$  reads  $\hat{u}_k(0) = \hat{f}(k)$ , and the solution is therefore

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) \exp(-c^2 k^2 t) \exp(i k x) dk.$$

- We can recognize this integral as the inverse Fourier transform of **a product of two Fourier transforms**,  $\hat{f}(k)$  and  $\hat{g}(k)$ , where  $\hat{g}(k) = \exp(-c^2 k^2 t)$ .

## Method of convolution

- Since we know that  $g(x) = \frac{1}{\sqrt{2c^2t}} \exp\left(-\frac{x^2}{4c^2t}\right)$ , and since the inverse Fourier transform of a product is the convolution of the inverse transforms times  $\frac{1}{\sqrt{2\pi}}$ , we therefore have

$$u(x, t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(y) \exp\left(-\frac{(x-y)^2}{4c^2t}\right) dy.$$

- **Example:** Solve the heat equation on the whole line with initial condition  $u(x, 0) = 1$  if  $|x| < 1$  and  $u(x, 0) = 0$  otherwise.
- Experiment with the *Heat Equation on the Whole Line* MATLAB GUI.