

Chapter 5: Expansions

Sections 5.1, 5.2, 5.7 & 5.8

1. Power series solutions of ordinary differential equations

- A **power series** about $x = x_0$ is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

- This series **is convergent** (or **converges**) if the sequence of partial sums

$$S_n(x) = \sum_{i=0}^n a_i(x - x_0)^i$$

has a (finite) limit, $S(x)$, as $n \rightarrow \infty$. In such a case, we write

$$S(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

- If the series is not convergent, we say that it is **divergent**, or that it **diverges**.

Radius of convergence

- One can show (Abel's lemma) that if a power series converges for $|x - x_0| = R_0$, then it converges absolutely for all x 's such that $|x - x_0| < R_0$.
- This allows us to define the **radius of convergence** R of the series as follows:
 - If the series only converges for $x = x_0$, then $R = 0$.
 - If the series converges for all values of x , then $R = \infty$.
 - Otherwise, R is the largest number such that the series converges for all x 's that satisfy $|x - x_0| < R$.

- A useful **test for convergence** is the **ratio test**:

$$R = \frac{1}{K}, \text{ where } K = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|,$$

where K could be infinite or zero, and it is assumed that the a_n 's are non-zero.

Power series as solutions to ODE's

- **Taylor series** are power series.
- A function f is **analytic** at a point $x = x_0$ if it can locally be written as a convergent power series, i.e. if there exists $R > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

for all x 's that satisfy $|x - x_0| < R$.

- If the functions p/h and q/h in the differential equation

$$h(x)y'' + p(x)y' + q(x) = 0 \quad (1)$$

are analytic at $x = x_0$, then every solution of (1) is analytic at $x = x_0$.

Power series as solutions to ODE's (continued)

- We can therefore look for solutions to (1) in the form of a power series.
- **Example:** Solve $y'' - 2y' + y = 0$ by the power series method.
- Many **special functions** are defined as power series solutions to differential equations like (1).
 - **Legendre polynomials** are solutions to Legendre's equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ where n is a non-negative integer.
 - **Bessel functions** are solutions to Bessel's equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ with $\nu \in \mathbb{C}$.

Separation of variables for the wave equation

- Consider the wave equation on a string of length L :

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

for a function $u(x, t)$ defined on a rectangle $[0, L] \times [0, T_f]$.

- We may try to solve the equation first by assuming $u(x, t) = X(x)T(t)$, where X and T are functions of one variable. Plugging this into the equation we get

$$X(x)T''(t) = X''(x)T(t),$$

or

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Wave equation, cont'd

- Since one side is a function of t alone and the other is a function of x alone, we find that both must be equal to some constant, i.e.,

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = k$$

for some constant k . (We don't know what k is!)

- If we fix the endpoints of the string, say $u(0, t) = u(L, t) = 0$, then in order to solve the wave equation, we must solve the following boundary value problem:

$$X'' - kX = 0, \quad X(0) = X(L) = 0.$$

for different values of k .

- This is a Sturm-Liouville problem.

2. Sturm-Liouville problems

- A **regular Sturm-Liouville problem** is an eigenvalue problem of the form

$$Ly = -\lambda \sigma(x)y, \quad Ly = [p(x)y']' + q(x)y, \quad (2)$$

p , q and σ are real continuous functions on $[a, b]$, $a, b \in \mathbb{R}$, $p(x) > 0$ and $\sigma(x) > 0$ on $[a, b]$, and $y(x)$ is square-integrable on $[a, b]$ and satisfies given **boundary conditions**.

- In what follows, we will use **separated boundary conditions**

$$C_1y(a) + C_2y'(a) = 0, \quad C_3y(b) + C_4y'(b) = 0. \quad (3)$$

- An **eigenvalue** of the Sturm-Liouville problem is a number λ for which there exists an **eigenfunction** $y(x) \neq 0$ that satisfies (2) and (3).

Sturm-Liouville problems (continued)

- One can show that with separated boundary conditions, **all eigenvalues** of the Sturm-Liouville problem **are real** (assuming they exist).
- In such a case, eigenfunctions associated with different eigenvalues are **orthogonal** (with respect to the weight function σ).
- Two functions $y_1(x)$ and $y_2(x)$ are **orthogonal with respect to the weight function σ** ($\sigma(x) > 0$ on $[a, b]$) if

$$\langle y_1, y_2 \rangle \equiv \int_a^b y_1(x) y_2(x) \sigma(x) dx = 0.$$

Sturm-Liouville problems (continued)

- Legendre's and Bessel's equations are examples of **singular Sturm-Liouville** problems.
- Legendre's equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ can be written as

$$[p(x)y']' + q(x)y = -\lambda y$$

where $p(x) = 1 - x^2$, $q(x) = 0$ and $\lambda = n(n + 1)$. In this case there are no boundary conditions and $[a, b] = [-1, 1]$.

- Bessel's equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ can be written in the form (2) by setting $p(x) = \sigma(x) = x$, $\lambda = 1$, and $q(x) = -\nu^2/x$. In this case, $[a, b] = [0, R]$, $R > 0$ and $y(x)$ is required to vanish at $x = R$.

3. Orthogonal eigenfunction expansions

- Recall that if A is a square $n \times n$ matrix with real entries, then the (genuine and generalized) eigenvectors of A , U_1, U_2, \dots, U_n , form a **basis** of \mathbb{R}^n .
- This means that every vector $X \in \mathbb{R}^n$ can be written in the form

$$X = a_1 U_1 + a_2 U_2 + \dots + a_n U_n, \quad (4)$$
 where the **coefficients** a_i are **uniquely determined**.
- Moreover, if the U_i 's are orthonormal** (i.e. orthogonal and of norm one), then each coefficient a_i can be found by taking the dot product of X with U_i , i.e. $a_i = \langle X, U_i \rangle$.
- In this case, (4) is an **orthogonal expansion** of X on the eigenvectors of A .

Orthogonal eigenfunction expansions (continued)

- Similarly, there exist special linear differential operators, such as Sturm-Liouville operators, whose eigenfunctions form a **complete orthonormal basis** for a space of functions satisfying given boundary conditions.
- We can then use such a **complete orthonormal basis**, $\{y_1, y_2, \dots\}$, to write any function in the space as a uniquely determined linear combination of the basis functions. Such an expansion is called an **orthonormal expansion** or a **generalized Fourier series**.
- In such a case, for every function f in the space, we can write

$$f(x) = \sum_{i=1}^{\infty} a_i y_i(x), \quad a_i = \langle f, y_i \rangle, \quad \|y_i\| = 1.$$

Trigonometric series

- **Trigonometric series** are the most important example of Fourier series.
- Consider the Sturm-Liouville problem with **periodic boundary conditions** ($p(x) = 1$, $q(x) = 0$, $\sigma(x) = 1$),

$$y'' + \lambda y = 0, \quad y(\pi) = y(-\pi), \quad y'(\pi) = y'(-\pi).$$

- The eigenfunctions are 1 , $\cos(x)$, $\sin(x)$, $\cos(2x)$, $\sin(2x)$, \dots , $\cos(mx)$, $\sin(mx)$, \dots , and correspond to the eigenvalues 0 , 1 , 1 , 4 , 4 , \dots , m^2 , m^2 , \dots .
- The above eigenfunctions are **orthogonal but not of norm one**. They can be made **orthonormal** by dividing each eigenfunction by its norm. They form a **complete basis** of the space of square integrable functions on $[-\pi, \pi]$.