Chapter 11: Fourier Transforms
Sections 8 & 9
1. Fourier transforms

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The relation \( f = \mathcal{F}^{-1}(\mathcal{F}(f)) \) reads

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\nu) \exp(ik(x - \nu)) \, d\nu \, dk. \quad (1)
\]
Properties of the Fourier transform

- As for Fourier series, Equation (1), i.e. \( f(x) = \left( \mathcal{F}^{-1}(\hat{f}) \right)(x) \)
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- The Fourier transform is a linear transformation, i.e. if \( f_1 \) and \( f_2 \) are such that their Fourier transforms exist and if \( \alpha \) and \( \beta \) are two arbitrary constants, then

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- Fourier transform of the derivative. If \( f \) and its derivatives are piecewise continuously differentiable and are absolutely integrable on \( \mathbb{R} \), and if \( \lim_{x \to \pm \infty} f(x) = 0 \), then the Fourier transform of the derivative of \( f \) is such that \( \hat{f}'(k) = ik \hat{f}(k) \).
Convolution

- The convolution of two absolutely integrable functions \( f \) and \( g \) is denoted by \( f \ast g \) and defined as

\[
(f \ast g)(x) = \int_{-\infty}^{\infty} f(x - t)g(t) \, dt = \int_{-\infty}^{\infty} f(t)g(x - t) \, dt.
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The convolution of two absolutely integrable functions \( f \) and \( g \) is denoted by \( f * g \) and defined as

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Convolution theorem. If \( f \) and \( g \) are both piecewise continuously differentiable and absolutely integrable on \( \mathbb{R} \), then the Fourier transform of the convolution of \( f \) and \( g \) is given by

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\mathcal{F}(f * g) = \sqrt{2\pi} \, \mathcal{F}(f) \, \mathcal{F}(g).
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Convolution theorem. If $f$ and $g$ are both piecewise continuously differentiable and absolutely integrable on $\mathbb{R}$, then the Fourier transform of the convolution of $f$ and $g$ is given by

$$\mathcal{F}(f \ast g) = \sqrt{2\pi} \mathcal{F}(f) \mathcal{F}(g).$$

Example: Find the Fourier transform of $f \ast g$ where $f(x) = \exp(-ax^2)$, $a > 0$, and $g$ is such that $g(x) = \exp(-ax)$ if $x > 0$ and $g(x) = 0$ otherwise.
2. Sine and cosine transforms

Consider a piecewise continuously differentiable function $f$, which is absolutely integrable on $\mathbb{R}$.

- **If $f$ is even**, then the Fourier transform of $f$ can be written as a cosine transform, i.e.
  \[
  \hat{f}(k) = \hat{f}_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(kx) \, dx,
  \]
  and
  \[
  f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(k) \cos(kx) \, dk.
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f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(k) \cos(kx) \, dk.
$$

- **Similarly, if $f$ is odd**, then the Fourier transform of $f$ is a sine transform, i.e. $\hat{f}(k) = -i \hat{f}_s(k)$, where

$$
\hat{f}_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(kx) \, dx, \quad f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(k) \sin(kx) \, dk.
$$