Chapter 6: Laplace Transforms

1. Definitions

- The Laplace transform, $\mathcal{L}(f)$, of a piecewise continuous function f (defined on $[0, \infty)$) is given by

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty \exp(-s\,t)\, f(t)\; dt.$$

- Clearly, the above integral only converges if f does not grow too fast at infinity. More precisely, if there exist constants $M > 0$ and $k \in \mathbb{R}$ such that

$$|f(t)| \leq M \exp(k\,t)$$

  for t large enough, then the Laplace transform of f exists for all $s > k$.

- If f has a Laplace transform F, we also say that f is the inverse Laplace transform of F, and write $f = \mathcal{L}^{-1}(F)$.

2. Properties of the Laplace transform

- The Laplace transform is a linear transformation, i.e. if f_1 and f_2 have Laplace transforms, and if α_1 and α_2 are constants, then

$$\mathcal{L}\left(\alpha_1 f_1 + \alpha_2 f_2\right) = \alpha_1 \mathcal{L}(f_1) + \alpha_2 \mathcal{L}(f_2).$$

- As for Fourier transforms, the statement

$$f = \mathcal{L}^{-1}\left(\mathcal{L}(f)\right)$$

  should be understood in a point-wise fashion only at points where f is continuous.

- Since there is no explicit formula for the inverse Laplace transform, formal inversion is accomplished by using tables, shifting t and s, taking derivatives of known Laplace transforms, or integrating them.

s-shifting, Laplace transform of derivatives & antiderivatives

- Note: All of the formulas written in what follows implicitly assume that the various functions used have well-defined Laplace transforms. One should therefore check that the corresponding Laplace transforms exist before using these formulas.

- s-shifting formulas

$$\mathcal{L}\left(e^{a\,t} f(t)\right)(s) = F(s-a), \qquad e^{a\,t} f(t) = \mathcal{L}^{-1}\left(F(s-a)\right)(t).$$

- Laplace transform of derivatives

$$\begin{aligned}\mathcal{L}\left(f'\right)(s) &= s\,\mathcal{L}(f)(s) - f(0),\\ \mathcal{L}\left(f''\right)(s) &= s^2\mathcal{L}(f)(s) - s\,f(0) - f'(0).\end{aligned}$$

Laplace transform of derivatives and antiderivatives

- More generally,

$$\mathcal{L}\left(f^{(n)}\right)(s) = s^n\mathcal{L}(f)(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

- Laplace transform of antiderivatives

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)(s) = \frac{1}{s}\,\mathcal{L}(f)(s),$$
$$\int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left(\frac{1}{s}\,\mathcal{L}(f)(s)\right)(t).$$

- **Examples:**
  - Find the Laplace transforms of $\sin(\omega t)$ and $\cos(\omega t)$.
  - Find the inverse Laplace transforms of $1/(s(s^2+1))$ and $1/(s^2(s^2+1))$.

Heaviside and delta functions; t-shifting

- The Heaviside function (or step function) $H(t)$ is defined as

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}.$$

- We can calculate that, for $a > 0$, $\mathcal{L}\left(H(t-a)\right)(s) = \dfrac{e^{-as}}{s}$.
- More generally, we have the following time-shifting formulas for $a > 0$.

$$\mathcal{L}\left(f(t-a)\,H(t-a)\right)(s) = e^{-as}\mathcal{L}(f)(s)$$
$$f(t-a)\,H(t-a) = \mathcal{L}^{-1}\left(e^{-as}\mathcal{L}(f)(s)\right)(t).$$

- The above formulas are useful to calculate the Laplace transforms of signals that are defined in a piecewise fashion.

Delta functions

- The Dirac delta function (or distribution) is defined as the limit of the following sequence of narrow top-hat functions,

$$\delta(t) = \lim_{\epsilon \longrightarrow 0} f_\epsilon(t), \qquad f_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon} & \text{if } |t| \le \epsilon \\ 0 & \text{otherwise} \end{cases}.$$

- Since $\int_{-\infty}^{\infty} f_\epsilon(t)\ dt = 1$, we also write that $\int_{-\infty}^{\infty} \delta(t)\ dt = 1$.
- More generally, for a "well-behaved" function g, we have $\int_{-\infty}^{\infty} g(t)\ \delta(t-a)\ dt = g(a)$.
- For $a > 0$, this allows us to define the Laplace transform of $\delta(t-a)$ as

$$\mathcal{L}\left(\delta(t-a)\right)(s) = e^{-as}.$$

Differentiation and integration of Laplace transforms

In what follows, we write $\mathcal{L}(f)(s)$ as $F(s)$.

- Differentiation of Laplace transforms

$$\mathcal{L}\left(t\, f(t)\right)(s) = -F'(s), \qquad \mathcal{L}^{-1}\left(F'(s)\right)(t) = -t\, f(t).$$

- Integration of Laplace transforms

$$\mathcal{L}\left(\frac{f(t)}{t}\right)(s) = \int_s^{\infty} F(\nu)\ d\nu,$$
$$\mathcal{L}^{-1}\left(\int_s^{\infty} F(\nu)\ d\nu\right)(t) = \frac{f(t)}{t}.$$

- **Example:** Find the inverse Laplace transform of $s/(s^2+1)^2$.

Applications to ODEs and systems of ODEs

- Solve $y'' + y = t/\pi$, with $y(\pi) = 0$ and $y'(\pi) = 1 + 1/\pi$.
- Let $f(t) = \begin{cases} \frac{1}{2\epsilon} & \text{if } 1-\epsilon \leq t \leq 1+\epsilon \\ 0 & \text{otherwise} \end{cases}$, where $\epsilon < 1$. Solve $y'' + 4y' - 5y = f(t)$ with initial conditions $y(0) = 0$ and $y'(0) = 0$.
- Solve $y'' + 4y' - 5y = \delta(t-1)$, with initial conditions $y(0) = 0$, $y'(0) = 0$.
- Solve the initial value problem $\frac{dX}{dt} = AX$,

$$A = \begin{bmatrix} -13 & -36 \\ 6 & 17 \end{bmatrix}, \qquad X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$