1.(15pts)

a. (10pts) Compute the determinant of the following matrix:

$$A = \begin{pmatrix} a & a & 0 & 0 \\ 0 & a & a & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & a \end{pmatrix}$$

Good, Good, NOT Good (want to zero this term!)

Gaussian Elimination

$$\xrightarrow[-\left(\frac{b}{a}\right) \text{Row } 1]{\text{Row } 4} \begin{pmatrix} a & a & 0 & 0 \\ 0 & a & a & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a - b\cdot\left(\frac{b}{a}\right) \end{pmatrix} = B.$$

~~[illegible] doesn't~~

$$\det(A) = \det(B) = a \cdot a \cdot a \cdot \left[a - b\left(\frac{b}{a}\right)\right] = a^3 \cdot \frac{a^2-b^2}{a}$$

$$= a^2(a^2-b^2)$$

$$= a^4 - a^2b^2$$

if all the element below diagonal are zeros. then the determinant can be calculated by multiplying the diagonal elements.

b. (5pts) Find the conditions on a and b so that the matrix does NOT have an inverse.

when $\det(A) = 0$, the matrix A doesn't have an inverse.

so, the condition is.

$$\det(A) = a^4 - a^2b^2 = 0$$

$$\Rightarrow a^2(a^2-b^2) = 0$$

$$\Rightarrow a^2(a+b)(a-b) = 0$$

$$\Rightarrow a = 0, \text{ or } a = -b, \text{ or } a = b$$

2. (20pts) Find the general solution of the following linear system of ODE:

$$\frac{dy_1}{dx} = y_1 + y_2$$
$$\frac{dy_2}{dx} = 2y_1.$$

Write your answer in the form $y_1(x) = \cdots$, $y_2(x) = \cdots$.

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 2 & 0-\lambda \end{pmatrix} = (1-\lambda)(-\lambda) - 2 = \lambda^2 - \lambda - 2 = 0$$

$$(\lambda-2)(\lambda+1) = 0$$

$$\lambda = 2 \qquad \lambda = -1$$

For $\lambda = 2$. $\begin{pmatrix} 1-2 & 1 \\ 2 & 0-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\begin{cases} -x_1 + x_2 = 0 \\ 2x_1 - 2x_2 = 0 \end{cases} \Rightarrow x_1 = x_2 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = 2$

For $\lambda = -1$ $\begin{pmatrix} 1+1 & 1 \\ 2 & 0+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\begin{cases} 2x_1 + x_2 = 0 \\ 2x_1 + x_2 = 0 \end{cases} \Rightarrow x_2 = -2x_1 \Rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector for $\lambda = -1$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{2x}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-x}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \begin{aligned} y_1(x) &= c_1 e^{2x} + c_2 e^{-x} \\ y_2(x) &= c_1 e^{2x} - 2c_2 e^{-x} \end{aligned}$$

3. (25pts)

a. (20pts) Find the general solution of the following linear ODE:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x.$$

Solve the homogeneous eqn first,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda+1)^2 = 0$$

$$\lambda = -1, \quad \lambda = -1 \quad \text{(double root)}$$

$$y_h = c_1 e^{-x} + c_2 x e^{-x}$$

Then find a particular solution, (Guess it!)

$$y_p = Ax + B.$$

$$y_p' = A \quad y_p'' = 0$$

Group

Compare with the given RHS

$$y_p'' + 2y_p' + y_p = 0 + 2A + Ax + B = (2A+B) + Ax \equiv x$$

$$\text{So, } \begin{cases} A = 1 \\ 2A + B = 0 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -2 \end{cases} \Rightarrow y_p = x - 2$$

So, $y = y_h + y_p = c_1 e^{-x} + c_2 x e^{-x} + x - 2$

b. (5pts) Find the particular solution of the ODE above with the initial condition $y(0) = 1$, $y'(0) = 1/2$.

Now, since ~~the~~ the initial condition is given. we use it to ~~~~ determine the constants c_1 and c_2.

$$\boxed{y(0) = c_1 + 0 + 0 - 2} = 1$$

$$y'(x) = -c_1 e^{-x} + c_2 e^{-x} - c_2 x e^{-x} + 1$$

$$\boxed{y'(0) = -c_1 + c_2 + 1} = \frac{1}{2}$$

$$\Rightarrow \begin{cases} c_1 - 2 = 1 \\ -c_1 + c_2 + 1 = \frac{1}{2} \end{cases} \Rightarrow \begin{array}{l} c_1 = 3 \\ c_2 = \frac{1}{2} - 1 + c_1 = \frac{1}{2} - 1 + 3 = \frac{5}{2} \end{array}$$

So, $y = 3e^{-x} + \frac{5}{2} x e^{-x} + x - 2$ is the particular solution that satisfy the initial cond.

4. **(20pts)**

a. (10pts) Write the following differential equation as a first order system:

$$\frac{d^3y}{dx^3} + x^2\frac{dy}{dx} - y^3x = 1.$$

Express your answer as $Y_0' = \cdots$, $Y_1' = \cdots$, etc.

Let $\begin{cases} Y_0 = y \\ Y_1 = y' \\ Y_2 = y'' \end{cases}$

$$\frac{d}{dx}\begin{pmatrix} Y_0 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix}$$

$y''' \Updownarrow 1 - x^2y' + y^3x$

$\longrightarrow$ Rewrite the Right-hand-side with the correct notation.

$$\frac{d}{dx}\begin{pmatrix} Y_0 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ 1 + Y_0^3 x - x^2 Y_2 \end{pmatrix}$$

b. (10 pts) For the following, circle ALL statements about the above differential equation which are necessarily true:

(i) The equation is homogeneous.

(ii) The equation is linear.

(iii) The equation, together with initial condition $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$ has a unique solution for x near 0.

(iv) The equation, together with initial condition $y(0) = 1$, $y'(0) = 0$ has a unique solution for x near 0.

(v) The equation, together with initial condition $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$ has a unique solution for all $x \in [-100, 100]$.

(vi) The equation, together with initial condition $y(0) = 1$, $y'(0) = 0$ has a unique solution for all $x \in [-100, 100]$.

5. **(20pts)** Find the first 3 terms of the power series solution of the following initial value problem:

$$\frac{dy}{dx} = y - x$$
$$y(0) = 1$$

$y = a_0 + a_1 x + a_2 x^2 + \cdots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$y - x = a_0 + (a_1 - 1)x + a_2 x^2 + \cdots$

$\boxed{y' = y - x} \Rightarrow a_1 + 2a_2 x + 3a_3 x^2 + \cdots = a_0 + (a_1-1)x + a_2 x^2 + \cdots$

$\Rightarrow \begin{cases} a_1 = a_0 \\ 2a_2 = a_1 - 1 \\ 3a_3 = a_2 \\ 4a_4 = a_3 \\ \vdots \end{cases}$ $\Rightarrow$ solve. (use a_0 as a free parameter) $\begin{cases} a_1 = a_0 \\ a_2 = \frac{a_1 - 1}{2} = \frac{a_0 - 1}{2} \\ a_3 = \frac{a_2}{3} = \frac{a_0 - 1}{6} \\ \vdots \end{cases}$

$\Rightarrow y = \underbrace{a_0}_{a_0} + \underbrace{a_0 x}_{a_1 x} + \underbrace{\frac{a_0 - 1}{2} x^2}_{a_2 x^2} + \cdots$

is the general form of solution.

$y(0) = a_0 + 0 + 0 + 0 + \cdots$ (Although we don't know all the terms in the series, we still know for sure that if we plug in $x = 0$, $y = a_0$ is the only term left, and all others vanish.)

$= a_0 = 1$

So. $y = 1 + x + 0x^2 + \cdots$

6. (20pts) For the following Sturm-Liouville problem, find all POSITIVE eigenvalues λ together with corresponding eigenfunctions.

$\Downarrow$

$\lambda > 0$

$$y'' = -\lambda y$$
$$y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0$$

$$y'' + \lambda y = 0$$

$$r^2 + \lambda = 0 \quad \Rightarrow r^2 = -\lambda \qquad (\text{note that } -\lambda < 0)$$

so, $r = \pm\sqrt{\lambda}\, i$

Thus, the general form of solution is

$$y = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$$

$$y(0) = C_1 = 0 \quad \Rightarrow \quad C_1 = 0$$

$$\cancel{y\left(\frac{\pi}{2}\right) = 0}$$

$$y\left(\frac{\pi}{2}\right) = C_1 \cos\left(\sqrt{\lambda}\,\frac{\pi}{2}\right) + C_2 \sin\left(\sqrt{\lambda}\,\frac{\pi}{2}\right) = 0$$

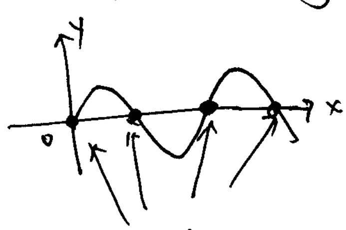
plug in $C_1 = 0$,

we will get $C_2 \sin\left(\sqrt{\lambda}\,\frac{\pi}{2}\right) = 0$

We don't want $C_2 = 0$, because if $C_2 = 0$,

then $y(x) \equiv 0$ (since we already know $C_1 = 0$)

so, our only option is $\sin\left(\sqrt{\lambda}\,\frac{\pi}{2}\right) = 0$



$x = n\pi$

$(n = 0, \pm 1, \pm 2, \cdots)$

$$\Rightarrow \sqrt{\lambda}\,\frac{\pi}{2} = n\pi$$

$$\Rightarrow \sqrt{\lambda} = 2n$$

$$\Rightarrow \boxed{\lambda_n = (2n)^2 = 4n^2}$$

eigenvalues

$$\boxed{y_n = C_2 \sin(2nx)}$$

eigenfunctions

7. (20 pts)

a. (10pts) Find the radius of convergence of the following power series:

$$\sum_{n=0}^{\infty} (n+1)\, x^n.$$

$a_n = n+1$

$$K = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n+2}{n+1} \right| = 1$$

$$R = \frac{1}{K} = 1$$

a'. (5pts extra credit) For which function is it a Taylor series?

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \Rightarrow \sum_{n=1}^{\infty} n (x)^{n-1} = \left(\frac{1}{1-x}\right)'$$

$$\sum_{n=0}^{\infty} (n+1) x^n = \left(\frac{1}{1-x}\right)' = \boxed{\frac{1}{(1-x)^2}}$$

b. (10pts) Give an example of three functions which are linearly independent. Show they are linearly independent.

$x, \quad x^2, \quad x^3$

or $\sin x \quad \cos x \quad \sin 2x$

$\vdots$

$x \quad x^2$