Final Exam Problems

1. Consider the matrix

\[
A = \begin{pmatrix}
1 & 2 & -3 \\
2 & 5 & 1 \\
3 & 7 & -2
\end{pmatrix}
\]

(a) Find all solutions to the equation \( Ax = 0 \).

Answer: Using row reduction we get

\[
\begin{pmatrix}
1 & 2 & -3 \\
2 & 5 & 1 \\
3 & 7 & -2
\end{pmatrix}
\]

and so we get

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= t \begin{pmatrix}
17t \\
-7t \\
t
\end{pmatrix}
\]

(b) Find all solutions (if any) to \( Ax = b \) where

\[
b = \begin{pmatrix}
2 \\
0 \\
2
\end{pmatrix}
\]

Answer: Do the same with the augmented matrix

\[
\begin{pmatrix}
1 & 2 & -3 & 2 \\
2 & 5 & 1 & 0 \\
3 & 7 & -2 & 2
\end{pmatrix}
\]

and so there is a solution,

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= t \begin{pmatrix}
17 \\
-7 \\
1
\end{pmatrix}
\]
c. Find all solutions (if any) to \(Ax = b\) where
\[
b = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.
\]

Answer: Do the same with the augmented matrix
\[
\begin{pmatrix} 1 & 2 & -3 & 2 \\ 2 & 5 & 1 & 0 \\ 3 & 7 & -2 & 1 \\ 1 & 2 & -3 & 2 \\ 0 & 1 & 7 & -4 \\ 0 & 1 & 7 & -5 \\ 1 & 2 & -3 & 2 \\ 0 & 1 & 7 & -4 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\]
so there are no solutions.

2.

a. Find all the eigenvalues and eigenvectors of the matrix \(A = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}\).

Answer: The Eigenvalues are \(-1\) and \(-6\) corresponding to Eigenvectors \(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\) and \(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\) respectively.

b. Find the general solution of the homogeneous ODE
\[
\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.
\]

Answer: Using part a, we see that the general solution is
\[
\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}
\]

3.

a. Write the following differential equation as a first order system:
\[
\frac{d^3 y}{dx^3} + x^2 \frac{dy}{dx} - y^3 = 0.
\]

Express your answer as \(Y_0' = \cdots\), \(Y_1' = \cdots\), etc.
Answer: We let $Y_0 = y$, $Y_1 = y'$, $Y_2 = y''$ and so

$$
Y_0' = Y_1 \\
Y_1' = Y_2 \\
Y_2' = -x^3Y_1 + Y_0^3
$$

b. For the following, circle ALL statements about the above differential equation which are necessarily true:

(i) The equation is homogeneous.

Answer: Yes, it is homogeneous. Each term has a $y$.

(ii) The equation is linear.

Answer: No, it is not linear because of the $y^3$ term.

4.

a. Find the Fourier series of the periodic function

$$
f(x) = \begin{cases} 
1 & \text{if } -1 < x \leq 0 \\
-2 & \text{if } 0 < x \leq 1 
\end{cases}
$$

extended to have period 2. It may help to know that \( \cos(\pi n) = (-1)^n \) and \( \sin(\pi n) = 0 \) for any integer \( n \).

Answer: We compute

$$
a_0 = \frac{1}{2} \int_{-1}^{1} f(x) \, dx = -1 \\

a_n = \int_{-1}^{0} \cos(\pi n x) \, dx - 2 \int_{0}^{1} \cos(\pi n x) \, dx = 0 \\
b_n = \int_{-1}^{0} \sin(\pi n x) \, dx - 2 \int_{0}^{1} \sin(\pi n x) \, dx \\
= \frac{-1 - (-1)^n}{n\pi} - 2 \frac{(-1)^n - 1}{n\pi} = \frac{3}{n\pi} (-1 - (-1)^n)
$$

So the series is

$$
-1 + \sum_{n=1}^{\infty} \frac{3}{n\pi} (-1 - (-1)^n) \sin n\pi x
$$

b. At which points in the interval $-1 \leq x \leq 1$ does the series not converge to the value of the function? To which values does the series converge at these points?

Answer: At all of these points, the series converges to $\frac{1 - 2}{2} = -\frac{1}{2}$.

5.
a. Consider the following partial differential equation for \( u(x, y) \):

\[
\begin{align*}
  u_{xx} + u_{yy} - 3u_y &= 0, \\
  u(x, 0) &= 0, \\
  u(x, 1) &= 0, \\
  u(0, y) &= 0, \\
  u(2, y) &= \sin(4\pi y).
\end{align*}
\]

Use separation of variables to obtain two ODE associated with this PDE. (DO NOT solve them.)

Answer: We consider 
\[
  u(x, y) = X(x)Y(y).
\]

The differential equation is

\[
X''Y + XY'' - 2XY' = 0
\]

so we get

\[
\frac{X''}{X} = -\frac{Y'' + 2Y'}{Y} = k
\]

for any constant \( k \). This gives the two ODE

\[
\begin{align*}
  X'' &= kX, \\
  Y'' - 2Y' &= -kY.
\end{align*}
\]

b. One (and only one) of those ODE’s can be formed into a Sturm-Liouville equation with homogeneous boundary conditions using the boundary conditions from the PDE. State which one and give the boundary conditions.

Answer: The only one that gives a full set of boundary conditions is the one for \( Y \), which gives \( Y(0) = 0 \) and \( Y(1) = 1 \). Note that you do get \( X(0) = 0 \) but cannot get another boundary condition for \( X \).

6. Consider the wave equation

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

for \( u(x, t) \) on a finite domain \( 0 \leq x \leq 10 \), with boundary conditions

\[
\begin{align*}
  u(0, t) &= 0, \\
  u(10, t) &= 0.
\end{align*}
\]

Recall that the general solution of the wave equation with these boundary conditions is of the form

\[
u(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi ct}{10} + B_n \sin \frac{n\pi ct}{10} \right) \sin \frac{n\pi x}{10}.
\]
If the equation is given initial conditions
\[ u(x, 0) = \sin(5\pi x), \]
\[ \frac{\partial u}{\partial t}(x, 0) = \sin(\pi x), \]
then find the particular solution for \( u(x, t) \) (that is, find the coefficients \( A_n \) and \( B_n \)). Note: you may leave your answer with terms like \( A_n = \sin \frac{3\pi n}{10} + \cos \pi n \) without further simplifying.

Answer: The two equations give
\[ \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{10} = \sin (5\pi x) \]
\[ \sum_{n=1}^{\infty} B_n \frac{n\pi c}{10} \sin \frac{n\pi x}{10} = \sin (\pi x). \]

It follows that all coefficients are zero except \( A_{50} = 1 \) and \( B_{10} = \frac{1}{\pi c} \). Hence the solution is
\[ u(x, t) = \frac{1}{\pi c} \sin \pi ct \sin \pi x + \cos 5\pi ct \sin 5\pi x. \]

7.
Consider the wave equation
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \]
\[ u(x, 0) = f(x) \]
\[ \frac{\partial u}{\partial t}(x, 0) = 0 \]
on the WHOLE LINE \((x \in (-\infty, \infty))\). Find \( \hat{u}(w, t) \), the Fourier transform of the solution \( u(x, t) \). DO NOT solve for \( u(x, t) \) (only its Fourier transform). Note: the answer should contain \( \hat{f}(w) \).

Answer: Taking the Fourier transform, we get
\[ \frac{\partial^2 \hat{u}}{\partial t^2} = -c^2 w^2 \hat{u} \]
\[ \hat{u}(w, 0) = \hat{f}(w) \]
\[ \frac{\partial \hat{u}}{\partial t}(w, 0) = 0. \]

The first equation can be solved as
\[ \hat{u}(w, t) = A(w) \cos cwt + B(w) \sin cwt. \]
Plugging in the initial conditions, we get

$$\hat{u}(w, t) = \hat{f}(w) \cos cwt.$$ 

8. For the following Sturm-Liouville problem, find all POSITIVE eigenvalues $\lambda$ together with corresponding eigenfunctions.

$$y'' = -\lambda y$$

$$y'(0) = 0, \quad y'(2) = 0$$

Answer: The solutions are

$$y = a \cos \sqrt{\lambda} x + b \sin \sqrt{\lambda} x.$$ 

Since

$$y' = -a \sqrt{\lambda} \sin \sqrt{\lambda} x + b \sqrt{\lambda} \cos \sqrt{\lambda} x$$

we can use the initial conditions to conclude that

$$b = 0$$

$$\sin 2\sqrt{\lambda} = 0$$

and so

$$2\sqrt{\lambda} = \pi n$$

$$\lambda = \left(\frac{n\pi}{2}\right)^2$$

are the eigenvalues and the corresponding eigenfunctions are

$$y_n = \cos \frac{n\pi}{2} x.$$ 

9. Consider the heat equation

$$u_t = c^2 u_{xx}.$$ 

We want to solve this equation for $0 < x < 1$ and for all $t$ with boundary conditions

$$u_x(0, t) = u_x(1, t) = 0$$

(Notice the derivative in the boundary conditions) and with the initial condition

$$u(x, 0) = x.$$ 

a. Separate the variables $u(x, t) = F(x)G(t)$ and find 2 ordinary differential equations satisfied by $F$ and $G.$
Answer: Using separation of variables, we get
\[ FG' = c^2 F''G \]
and so
\[ \frac{G'}{c^2G} = \frac{F''}{F} \]
and so this is equal to a constant and we get the two ODEs
\[ F'' = kF \]
\[ G' = c^2kG. \]

b. Using the boundary conditions, find the \( F_n \).
Answer: The boundary conditions become
\[ F'(0) = 0, \quad F'(1) = 0 \]
and so we get
\[ F = a \cos \sqrt{-k}x + b \sin \sqrt{-k}x \]
(positive \( k \) do not result in any solutions). The boundary conditions give \( b = 0 \) and \( \sin \sqrt{-k} = 0 \) and so \( k = -(\pi n)^2 \) for integers \( n \). It follows that
\[ F_n = \cos \pi nx. \]
(Note: \( n = 0 \) is allowed, too).

c. Find the \( G_n \) and write down the eigenfunctions \( u_n \).
Answer:
\[ G'_n = -c^2 \pi^2 n^2 G_n \]
and so
\[ G_n = e^{-c^2 \pi^2 n^2 t}. \]

d. Write down the general solution \( u(x,t) \) and use the initial condition to find the coefficients.
Answer: The general solution is
\[ u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-c^2 \pi^2 n^2 t} \cos \pi nx. \]
The initial condition gives that
\[ x = a_0 + \sum_{n=1}^{\infty} a_n \cos \pi nx. \]
We need to compute the coefficients of the even periodic extension, so we get
\[ a_0 = \int_{0}^{1} x dx = \frac{1}{2} \]
\[ a_n = 2 \int_{0}^{1} x \cos n\pi x dx = \frac{1}{\pi^2 n^2} \cos \pi n - 1 \]
\[ = \frac{1}{\pi^2 n^2} ((-1)^n - 1) \]

so the solution is

\[ u(x, t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{((-1)^n - 1)}{\pi^2 n^2} e^{-\frac{c^2 \pi^2 n^2 t}{\pi^2}} \cos \pi nx. \]

e. What is \( \lim_{t \to \infty} u(x, t) \)

Answer: As \( t \to 0 \), all of the coefficients go to zero (because of the exponential), and so we get the limit is \( \frac{1}{2} \).