Math 322. Spring 2015
Review Problems for Midterm 2

Linear Algebra:
Topic: Linear Independence of vectors.

Question 1.
Explain why if $A$ is not square, then either the row vectors or the column vectors of $A$ are linearly dependent.

Question 2.
Are the following vectors linearly independent: $[1 \ 2 \ 3 \ 4]$, $[2 \ 3 \ 4 \ 5]$, $[3 \ 4 \ 5 \ 6]$, $[4 \ 5 \ 6 \ 7]$?

Question 3.
Are the following vectors linearly independent: $[3 \ 4 \ 7]$, $[2 \ 0 \ 3]$, $[8 \ 2 \ 3]$, $[5 \ 5 \ 6]$?

Topic: Linear system of equations, rank, row space, column space, basis

Question 4.
Find the rank, a basis for the row space, and a basis for the column space of the following matrix.

$$
\begin{pmatrix}
8 & 2 & 5 \\
16 & 6 & 29 \\
4 & 0 & -7
\end{pmatrix}
$$

Question 5.
Let

$$
A = \begin{pmatrix}
-2 & 2 & 6 \\
1 & -1 & 2 \\
-1 & 1 & 3
\end{pmatrix}.
$$

Does the system $Ax = B$ with $B = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ admit solutions? If so, how many?

Question 6.
Does rank $A = \text{rank } B$ imply rank $A^2 = \text{rank } B^2$? If yes, justify your answer; otherwise, give a counterexample.

Question 7.
Let
\[ A = \begin{bmatrix} 0 & -6 & 4 \\ 1 & -2 & -2 \\ 1 & -8 & 2 \\ 3 & -12 & -2 \end{bmatrix} \]

(a) Find the rank of \( A \).
(b) Find a basis of the column space of \( A \).
(c) Find a basis of the row space of \( A \).
(d) Let \( x = [x_1, x_2, x_3]^T \). Find the general form of solutions for the homogeneous linear system of equations \( AX = 0 \).
(e) Find the dimension of the null space of \( A \).
(f) Let \( b = [1, 2, 3, 7]^T \). Does the following system of equations have solution(s)? If your answer is yes, find the general form of the solution(s).

\[ AX = b. \]

**Topic: Determinant.**

**Question 8:** Find the determinant of the following matrix.

\[ A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ -1 & 1 & 0 \end{pmatrix}. \]

**Question 9:** Let

\[ B = \begin{bmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & d & c \end{bmatrix} \]

Find \( \det(B) \).

**Topic: Eigenvectors and eigenvectors.**

**Question 10:**

Find the eigenvalues and eigenvectors of

\[ A = \begin{bmatrix} 2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \]

**Question 11:**

Find the eigenvalues and eigenvectors of
\[ B = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \]

**Understanding questions:**

**Question 12:** Is it possible for a linear system of equations to have exactly 10 solutions? Why or why not?

**Question 13:** Is it possible for a linear system of equations to have no solution at all? If so, give an example. If not, explain why.

**Question 14:** Give an example of a 3 by 3 matrix whose rank is 1. What is the dimension of the null space of the matrix you just found? Explain.

**Question 15:** Give an example of three 3-dimensional vectors that do not span \( \mathbb{R}^3 \). Choose the vectors so that no two vectors are proportional to one another.

**Question 16:** Give an example of three 3-dimensional vectors with non-zero entries that span \( \mathbb{R}^3 \).

**Question 17:** For a system of \( n \) equations with \( n \) unknowns of the form \( Ay = b \), list four ways to tell if there is a unique solution or not.

**Ordinary differential equations:**

**Topic: Linear independency of functions)**

**Question 1:**
Find an ODE for which the given functions \( y_1 = \cos x \) and \( y_2 = \sin x \) are solutions. Verify these two functions are linearly independent.

**Question 2:**
Find an ODE for which the given functions \( y_1 = e^x \) and \( y_2 = xe^x \) are solutions. Verify these two functions are linearly independent.

**Topic: Linear ODE (Existence and uniqueness of solutions, solving homogeneous linear ODE)**

**Question 3:** Find the general form of solution to the following equation.

\[ \frac{d^3y}{dx^3} - \frac{dy}{dx} = 0 \]

**Question 4:** Consider the following initial value problem.

\[ \frac{d^2y}{dx^2} - 4y = 0 \]
\[ y(0) = 1 \]
\[ y'(0) = 0. \]
(a). Does this initial value problem have a solution? Is the solution unique?

(b). Find the solution to this initial value problem if your answer for part (a) is yes.

**Topic: Linear ODE system**

**Question 5:** Consider the following initial value problem.

\[
\frac{dy_1}{dt} = y_1 + 2y_2 \\
\frac{dy_2}{dt} = 5y_1 - 2y_2 \\
y_1(0) = 2 \\
y_2(0) = 9.
\]

(a). Does this initial value problem have a solution? Is the solution unique?

(b). Find the solution to this initial value problem if your answer for part (a) is yes.

**Question 6:** Let

\[
A = \begin{pmatrix}
3 & 0 & 2 \\
0 & -2 & 3 \\
0 & 5 & -4 
\end{pmatrix}
\]

and

\[
Y = \begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix}
\]

Find the general form of solution to the following system of equations.

\[
\frac{dY}{dt} = AY.
\]

**Question 7:** Consider again the ODE given in Question 3.

(a). Convert this problem into an first-order ODE system.

(b). Solve this ODE system, and compare the solution to the solution you found for Question 3.

**Understanding questions:**

**Question 8:** What can you say about the set of solutions to a homogeneous linear differential equation?

**Question 9:** Why do we look at the Wronskian to decide whether a set of functions is linearly independent?

**Question 10:** If the Wronskian of a set of functions is equal to zero for a particular value of x, does it necessarily mean that the functions are linearly dependent? Why or why not?

**Question 11:** What condition should the Wronskian of n functions satisfy for these functions to be linearly independent?

**Question 12:** If you are given a basis of solutions to a linear differential equation, do you know how to write down the general solution?