Math 322. Spring 2015
Review Problems for Midterm 3

Fourier Series:

Topic: Calculation of Fourier coefficients

Question 1.

Find the Fourier series of the following periodic function, of period $p = 2L = 2\pi$:

$$f(x) = 1 - 2\sin^2(2x).$$

Answer: First thing to notice is that this function is even,

$$f(-x) = f(x).$$

In fact

$$1 - 2\sin^2 2x = \cos^2 2x - \sin^2 2x = \cos 4x$$

so the Fourier series is $\cos 4x$.

Question 2.

Find the Fourier series of the periodic function, $p = 2L = 2$,

$$f(x) = \left\{ \begin{array}{cc} 0 & -1 < x < 0 \\ x & 0 < x < 1 \end{array} \right\}.$$

Answer: This function is neither odd nor even, so we need to compute all of the coefficients. They are

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{4} \\
a_n &= \int_{-1}^{1} f(x)\cos n\pi x dx \\
&= \left[\frac{1}{(n\pi)^2}\cos n\pi x + \frac{1}{n\pi}x\sin n\pi x\right]_0^1 \\
&= \frac{1}{(n\pi)^2}(\cos n\pi - 1) = \frac{(-1)^n - 1}{(n\pi)^2} \\
b_n &= \int_{-1}^{1} f(x)\sin n\pi x dx \\
&= \left[\frac{1}{(n\pi)^2}\sin n\pi x - \frac{1}{n\pi}x\cos n\pi x\right]_0^1 \\
&= -\frac{1}{n\pi}\cos n\pi = \frac{(-1)^{n+1}}{n\pi}
\end{aligned}$$

Thus we have that the series is

$$\frac{1}{4}+\sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{(n\pi)^2}\cos n\pi x+\frac{(-1)^{n+1}}{n\pi}\sin n\pi x\right).$$

Topic: Convergence of the Fourier series
Question 3.

For questions 1 and 2, describe what function the Fourier series represents on the whole real line.

Answer: For question 1, the function is periodic with period 2π and so the Fourier series converges to $f(x)$. For question 2, the Fourier series $F(x)$ will converge to $f(x)$ for $-1 < x < 1$ and then $F(-1) = F(1) = \frac{1}{2}$. The rest of $F(x)$ will be the periodic extension of these values. A very technical answer would give

$$F(x)=\begin{cases} f(x-2n) & \text{if } 2n-1<x<2n+1 \text{ for an integer } n \\ \frac{1}{2} & \text{if } x=2n+1 \text{ for an integer } n \end{cases}$$

Partial differential equations:

Topic: Sturm-Liouville problems

Question 1. For the following Sturm-Liouville problem, find the eigenvalues and eigenfunctions

$$y''+\lambda y=0,\quad y(0)=0,\quad y'(L)=0$$

Answer: We quickly see that the boundary conditions rule out exponential and linear solutions, so we must have

$$y=A\cos\sqrt{-\lambda}x+B\sin\sqrt{-\lambda}x$$

where $\lambda < 0$. The first initial condition gives that

$$A=0.$$

The second initial condition gives that

$$B\sqrt{-\lambda}\cos\sqrt{-\lambda}L=0.$$

Thus we get

$$\sqrt{-\lambda}L=\frac{\pi}{2}+\pi n$$

for integers n. Hence

$$\lambda=-\left(\frac{\pi}{L}\right)^2\left(n+\frac{1}{2}\right)^2$$

corresponding to eigenfunctions

$$\sin\left(\left(\frac{\pi}{L}\right)\left(n+\frac{1}{2}\right)x\right).$$

Topic: Wave equation

Question 2: For a string of length π, with endpoints fixed, the eigenfunctions obtained by separation of variables are given by

$$u_n(x,t) = G_n(t)F_n(x) \ = (A_n \cos(nct) + B_n \sin(nct)) \sin(nx).$$

If the initial deflection is zero, i.e. $f(x) = 0$, and the initial velocity is given by $g(x) = \sin(x)\cos(x)$, find the deflection $u(x,t)$. **Explain your work in detail**.

Answer: We consider the superposition

$$u(x,t) = \sum_{n=0}^{\infty} (A_n \cos(nct) + B_n \sin(nct)) \sin(nx)$$

and hence

$$\frac{\partial u}{\partial t}(x,t) == \sum_{n=0}^{\infty} (-ncA_n \sin(nct) + B_n nc \cos(nct)) \sin(nx)$$

And so

$$0 = \sum_{n=0}^{\infty} A_n \sin(nx)$$

and so $A_n = 0$. Also

$$\sin(x)\cos(x) = \sum_{n=0}^{\infty} B_n nc \sin(nx).$$

Since

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

We must have that all $B_n = 0$ except $B_2 = \frac{1}{4c}$. Thus

$$u(x,t) = \frac{1}{4c} \sin 2ct \sin 2x.$$

Topic: Heat equation

Question 3: A metal bar of length $L = \pi$ is perfectly insulated, including being perfectly insulated at the endpoints (the temperature is not fixed there). It turns out that the situation of no heat flux through the ends corresponds to the conditions $u_x(0,t) = 0$, $u_x(\pi,t) = 0$. Find all eigenfunctions for the heat equation in this case. Explain your steps in detail. **Note that you do not need to match initial conditions for this problem!**

Answer: We consider separation of variable, which leads to the Sturm-Liouville problem

$$X'' = kX$$
$$X'(0) = X'(\pi) = 0.$$

It follows that $k < 0$ and

$$X = A\cos\sqrt{-k}x + B\sin\sqrt{-k}x.$$

The boundary condition correspond to

$$X' = -A\sqrt{-k}\sin\sqrt{-k}x + B\sqrt{-k}\cos\sqrt{-k}x$$

and we find that the eigenfunctions for X are

$$X_n = \cos nx.$$

The corresponding problem for T is

$$T' = -n^2 T$$

and so we get

$$u_n\left(x, y\right) = e^{-n^2 t}\cos nx.$$