More problems on cardinality

1) Suppose A and B are countable sets. Show that $A \cap B$ is countable.

Answer: Since $A \cap B \subseteq A$, and any subset of a countable set is countable, $A \cap B$ must be countable.

2) Show that $S = \{2^n + 3^m : m, n \in \mathbb{N}\}$ is countable. (Hint: Don't try to use unique factorization, since there is a sum not a product!)

Answer: Consider the map $g : \mathbb{N} \times \mathbb{N} \to S$ given by $g(m, n) = 2^n + 3^m$. It is clearly surjective. Let $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection, then $g \circ f : \mathbb{N} \to S$ is a surjection, so, by our theorem from class, S is countable.

3) Suppose R is an equivalence relation on a countable set S, and let E be the collection of equivalence classes of R. Show that E is countable.

Answer: Define a function $f : E \to S$ by taking each equivalence class $e \in E$ to some element $s \in e$ (it doesn't matter which one, as long as we are consistent; since $e \neq \varnothing$, this function can be defined). Note that this function must be injective. Since S is countable, there is an injective map $S \to \mathbb{N}$, and thus by composing the two maps, we have an injective map $E \to \mathbb{N}$. Thus E is countable.