Practice problems on ordered fields

You may use any of the following (please use labeling below):
Ordered Field Axioms for set S with operations $+, \cdot$ and relation $<$.

A1. For all $x, y \in S$, $x+y \in S$ and if $x=w$ and $y=z$ then $x+y=w+z$.
A2. For all $x, y \in S$, $x+y=y+x$.
A3. For all $x, y, z \in S$, $x+(y+z)=(x+y)+z$
A4. There exists $0 \in S$ such that $x+0=x$ for all $x \in S$.
A5. For each $x \in S$, there exists a unique $-x \in S$ such that $x+(-x)=0$.
M1. For all $x, y \in S$, $xy \in S$ and if $x=w$ and $y=z$ then $xy=wz$.
M2. For all $x, y \in S$, $xy=yx$.
M3. For all $x, y, z \in S$, $x(yz)=(xy)z$
M4. There exists $1 \in S$ such that $1 \neq 0$ and $x \cdot 1=x$ for all $x \in S$.
M5. For each $x \in S \setminus \{0\}$, there exists a unique $\frac{1}{x} \in S$ such that $x \cdot \frac{1}{x}=1$.
DL. For all $x, y, z \in S$, $x(y+z)=xy+xz$.
O1. For all $x, y \in S$, exactly one of $x<y$, $x=0$, or $y<x$ holds.
O2. For all $x, y, z \in S$, if $x<y$ and $y<z$ then $x<z$.
O3. For all $x, y, z \in S$, if $x<y$ then $x+z<y+z$.
O4. For all $x, y, z \in S$, if $x<y$ and $z>0$ then $xz<yz$.

Theorem: Let $x, y, z \in S$. Then:

a. If $x+z=y+z$ then $x=y$.
b. $x \cdot 0=0$.
c. $(-1)x=-x$.
d. $xy=0$ if and only if $x=0$ or $y=0$.
e. $x<y$ if and only if $-y<-x$.
f. If $x<y$ and $z<0$ then $yz<xz$.

1) If $x \neq 0$ then $\frac{1}{x} \neq 0$ and $1/\left(\frac{1}{x}\right)=x$.

Answer: If $x \neq 0$, then axiom M5 says we have $\frac{1}{x}$ such that $x\frac{1}{x}=1$. By Theorem 11.1b, if $\frac{1}{x}=0$, then $x\left(\frac{1}{x}\right)=0$, but since $0 \neq 1$, this is a contradiction, so we must have that $\frac{1}{x} \neq 0$. Thus by axiom M5 again we must have an element $1/\left(\frac{1}{x}\right)$ that satisfies $\frac{1}{x}\left(1/\left(\frac{1}{x}\right)\right)=1$. Using axiom M1 and M4, we can see that

$$x\left(\frac{1}{x}\left(1/\left(\frac{1}{x}\right)\right)\right)=x \cdot 1=x.$$

Using associativity (M3) we have that

$$\begin{aligned} x\left(\frac{1}{x}\left(1/\left(\frac{1}{x}\right)\right)\right) &= \left(x\frac{1}{x}\right)\left(1/\left(\frac{1}{x}\right)\right) \\ &= 1\left(1/\left(\frac{1}{x}\right)\right) \\ &= 1/\left(\frac{1}{x}\right). \end{aligned}$$

We conclude that

$$1/\left(\frac{1}{x}\right) = x.$$

2) If $x \neq 0$, then $x^2 > 0$.

Answer: By O1, if $x \neq 0$ then either $x > 0$ or $x < 0$. If $x > 0$, we can use O4 to see that $x > 0$ implies $x^2 > x\,(0) = 0$ (using M4). If $x < 0$, then by Theorem part f, $x^2 > x \cdot 0 = 0$ (by Theorem part b).