Math 323: Homework 11 Solutions

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10.3)

Proposition 1 *For all $n \in \mathbb{N}$,*

$$1^2 + 2^3 + 3^2 + \cdots + n^2 = \frac{1}{6} n (n+1)(2n+1).$$

Proof. The basis is that $1^2 = 1 = \frac{1}{6} 1 (2)(2+1)$. Now suppose

$$1^2 + 2^3 + 3^2 + \cdots + k^2 = \frac{1}{6} k (k+1)(2k+1)$$

for some $k \in \mathbb{N}$. Then we see that

$$\begin{aligned} 1^2 + 2^3 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{1}{6} k (k+1)(2k+1) + (k+1)^2 \\ &= \frac{1}{6} (k+1) \left(2k^2 + k + 6k + 6\right) \\ &= \frac{1}{6} (k+1)(k+2)(2k+3) \left(2k^2 + k + 6k + 6\right). \end{aligned}$$

By the principle of mathematical induction, the proposition is proved. ■

10.6)

Proposition 2 *For all $n \in \mathbb{N}$,*

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Proof. The basis is that $\frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1}$. Now suppose

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

for some $k \in \mathbb{N}$. Then consider

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}.$$

It follows that

$$\begin{aligned} \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} &= \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2}. \end{aligned}$$

By the principle of mathematical induction, the proposition is proved. ■

10.13)

Proposition 3 *For all $n \in \mathbb{N}$, $5^{2n} - 1$ is a multiple of* 8.

Proof. The basis is that $5^2 - 1 = 24 = 3(8)$. Now suppose $5^{2k} - 1 = 8m$ for some $m \in \mathbb{N}$. Then consider

$$\begin{aligned} 5^{2(k+1)} - 1 &= 5^2 5^{2k} - 1 \\ &= 5^2 (8m + 1) - 1 \\ &= 8(25m + 3). \end{aligned}$$

Thus the proposition is proven by the principle of mathematical induction. ■

10.14)

Proposition 4 *For all $n \in \mathbb{N}$, $9^n - 4^n$ is a multiple of* 5.

Proof. The basis is that $9 - 4 = 5$. Suppose $9^k - 4^k = 5m$ for some $m \in \mathbb{N}$. Then $9^k = 4^k + 5m$. We now see that

$$\begin{aligned} 9^{k+1} - 4^{k+1} &= 9\left(9^k\right) - 4^{k+1} \\ &= 9\left(4^k + 5m\right) - 4^{k+1} \\ &= 5\left(4^k + 9m\right). \end{aligned}$$

The proposition is not proven by the principle of mathematical induction. ■

10.17)

Proposition 5 *For all $n \in \mathbb{N}$,*

$$5 + 9 + 13 + \cdots + (4n + 1) = n(2n + 3).$$

Proof. The basis is that $5 = 1(2(1) + 3)$. Now suppose $5 + 9 + \cdots + (4k + 1) = k(2k + 3)$. Then consider the sum

$$\begin{aligned} 5 + 9 + \cdots + (4k + 1) + 4(k + 1) + 1 &= k(2k + 3) + 4k + 5 \\ &= 2k^2 + 7k + 5 \\ &= (k + 1)(2(k + 1) + 3). \end{aligned}$$

Thus the proof is complete by the principle of mathematical induction. ■