Math 323: Homework 13 Solutions

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The ordering on the set of quotients of polynomials is the following. Consider an element

$$q = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_k x^k + \cdots + b_1 x + b_0}.$$

We say $q > 0$ if $a_n b_k > 0$. We then define

$$\frac{f}{g} > \frac{f'}{g'} \text{ iff } \frac{f}{g} - \frac{f'}{g'} > 0.$$

11.11a) We consider the order axioms. We consider

$$\begin{aligned} q &= \frac{f}{g} = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_k x^k + \cdots + b_1 x + b_0} \\ q' &= \frac{f'}{g'} = \frac{a'_{n'} x^{n'} + \cdots + a'_1 x + a'_0}{b'_{k'} x^{k'} + \cdots + b'_1 x + b'_0}. \end{aligned}$$

We see that

$$q - q' = \frac{fg' - f'g}{gg'}$$

and so the highest degree coefficient in the denominator is $b_k b'_{k'}$ and the highest degree coefficient in the numerator is either $a_n b'_{k'}$ or $-a'_{n'} b_k$ or $a_n b'_{k'} - a'_{n'} b_k$. In the three cases we have

$$\begin{aligned} b_k b'_{k'} a_n b'_{k'} &= a_n b_k \left(b'_{k'}\right)^2 \\ -b_k b'_{k'} a'_{n'} b_k &= -a_{n'} b'_{k'} \left(b_k\right)^2 \\ b_k b'_{k'} a_n b'_{k'} - b_k b'_{k'} a'_{n'} b_k &= a_n b_k \left(b'_{k'}\right)^2 - a_{n'} b'_{k'} \left(b_k\right)^2. \end{aligned}$$

For O1, since each of these can only be positive or negative or zero, we must have trichotomy. For O2 (transitivity), we see that if $q - q' > 0$ and $q' - q'' > 0$ then $q - q'' = q - q' + q' - q'' > 0$. Finally, for O3, we see that if $q - q' > 0$ then $q + r - (q' + r) = q - q' > 0$.

11.11b) We claim:

$$-x^3 < 3 - x < 5 < x + 2 < x^2$$

We see that $3 - x + x^3 > 0$, $5 - (3 - x) - x + 2 > 0$, $x + 2 - 5 = x - 3 > 0$, $x^2 - (x + 2) > 0$.

11.13a) Consider $(a, b) \sim (c, d)$ if $ad = bc$. We see that this is an equivalence relation:
reflexive: $(a, b) \sim (a, b)$ since $ab = ab$.
symmetric: If $(a, b) \sim (c, d)$ then $ad = bc$, but then $cb = da$ (using commutativity) so $(c, d) \sim (a, b)$
transitive: If $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ then we have $ad = bc$ and $cf = de$. We see that

$$afdc = bedc$$

and so we conclude that $af = be$ and $(a, b) \sim (e, f)$, or $c = 0$ (since $d \neq 0$). If $c = 0$, it follows that $a = 0$ and $e = 0$, so we still have $(a, b) \sim (e, f)$.

11.13b) Suppose $[a/b] = [p/q]$ and $[c/d] = [r/s]$, so $aq = bp$ and $cs = dr$. Since

$$\begin{aligned} [a/b] + [c/d] &= \left[\frac{ad+bc}{bd}\right] \\ [p/q] + [r/s] &= \left[\frac{ps+rq}{qs}\right], \end{aligned}$$

we note that

$$\begin{aligned} \frac{ad+bc}{bd} &= \frac{ad+bc}{bd}\frac{qs}{qs} \\ &= \frac{aqds+bqcs}{bdqs} \\ &= \frac{bpds+bqdr}{bdqs} \\ &= \frac{ps+qr}{qs}\frac{bd}{bd} \\ &= \frac{ps+qr}{qs}. \end{aligned}$$

For multiplication, we see that

$$\begin{aligned} [a/b]\,[c/d] &= \left[\frac{ac}{bd}\right] \\ [p/q]\,[r/s] &= \left[\frac{pr}{qs}\right] \end{aligned}$$

and we see that

$$\begin{aligned} \frac{ac}{bd} &= \frac{ac}{bd}\frac{qs}{qs} \\ &= \frac{aqcs}{bdqs} \\ &= \frac{bpdr}{bdqs} \\ &= \frac{pr}{qs}\frac{bd}{bd}. \end{aligned}$$

11.13c) We claim that $b \neq 0$ implies that $[0/b] = [0/1]$ and $[b/b] = [1/1]$. This is easy to check:

$$\begin{aligned} 0 \cdot 1 = 0 &= 0 \cdot b \\ b1 = b &= b1. \end{aligned}$$

11.13d) Let $a, b \in \mathbb{Z}$ and $b \neq 0$. We see that

$$\begin{aligned} [a/b] + [0/1] &= \left[\frac{a(1)+0(b)}{b(1)}\right] = [a/b] \\ [a/b]\,[1/1] &= \left[\frac{a(1)}{b(1)}\right] = [a/b]. \end{aligned}$$