Math 323: Homework 14 Solutions

David Glickenstein

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12.3l)

Proposition 1 *If* $S = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 2 - \frac{1}{n}\right]$, *S has no maximum and* $\sup S = 2$.

Proof. We note that 2 is not in $\left[\frac{1}{n}, 2 - \frac{1}{n}\right]$ for any n, so 2 is not in S, and so S cannot have a maximum (since it if did, the maximum would be the supremum). We now show that ■

$\sup S = 2$. We first note that for all $n \in \mathbb{N}$ and for all $x \in \left[\frac{1}{n}, 2 - \frac{1}{n}\right]$, $x \leq 2 - \frac{1}{n} < 2$, so 2 is an upper bound. Now suppose $y < 2$. Since $0 < 2 - y$, the Archimedean property gives us that there exists $m \in \mathbb{N}$ such that $0 < \frac{1}{m} < 2 - y$, and thus

$$y < y + \frac{1}{m} \in \left[\frac{1}{m}, 2 - \frac{1}{m}\right].$$

This completes the proof.

12.3n)

Proposition 2 *If* $S = \left\{r \in \mathbb{Q} : r^2 \leq 5\right\}$, *then S has no maximum and* $\sup S = \sqrt{5}$.

Proof. We know that since 5 is prime, $\sqrt{5}$ is not rational, so S cannot have a maximum. We now show that $\sup S = \sqrt{5}$ (note: we are assuming the existence of $\sqrt{5} \in \mathbb{R}$). Clearly, since $r^2 \leq 5$, we must have $r \leq \sqrt{5}$ if $r \in S$ (recall that this can easily be proven using the contrapositive). Thus $\sqrt{5}$ is an upper bound for S. Now suppose $s < \sqrt{5}$. The density of the rationals says there exists a rational number $r \in \mathbb{Q}$ such that $s < r < \sqrt{5}$, and so we are done. If we do not want to use the density of the rationals, we can do the following: $\sqrt{5} - s > 0$ so there is a natural number n such that $\frac{1}{n} < \sqrt{5} - s$. We would like to take $s + \frac{1}{n}$, but this may not be rational, so instead we see that ■

12.6a)

Proposition 3 *Let S be a nonempty bounded subset of* $\mathbb{R}$. *Then* $\sup S$ *is unique.*

Proof. Suppose x and y are both suprema of S. We wil show that $x = y$. Since x is the least upper bound and y is an upper bound, we must have that $x \leq y$. Similarly, since y is the least upper bound and x is an upper bound, we must have $y \leq x$. Thus $x = y$. ■

12.6b)

Proposition 4 *Suppose m and n are both maxima of a set S. Then* $m = n$.

Proof. If m and n are both maxima, then they are both suprema. By the previous part, they must be equal. ■