Math 323: Homework 4 Solutions

David Glickenstein

February 11, 2013

4.8)

Proposition 1 *If* $x/(x-1) \leq 2$, *then* $x < 1$ *or* $x \geq 2$.

Proof. We first note that $x/(x-1) \leq 2$ cannot be true if $x = 1$, so in this case the implication is definitely true. We then consider the cases $x - 1 < 0$ and $x - 1 > 0$ separately. In the first case, we see that $x/(x-1) \leq 2$ implies

$$\begin{aligned} x &\geq 2(x-1) \\ 2 &\geq x, \end{aligned}$$

which is always true if $x < 1$. In the second case, $x/(x-1) \leq 2$ implies that

$$\begin{aligned} x &\leq 2(x-1) \\ 2 &\leq x. \end{aligned}$$

So in this case, we have $x > 1$ and $x \geq 2$, so $x \geq 2$. Combining the cases, we find that $x < 1$ or $x \geq 2$. ■

Note, there are other ways to prove this. For instance, one could prove that if $x/(x-1) \leq 2$ and $x \geq 1$, then $x \geq 2$. Note that for $x/(x-1) \leq 2$ and $x \geq 1$ to be true, we actually must have $x > 1$. Another possibility is to prove the contrapositive, which would prove that if $1 \leq x < 2$ then $x/(x-1) > 2$ or $x = 1$ (note the negation of $x/(x-1) \leq 2$ is not $x/(x-1) > 2$, but $x/(x-1) > 2$ or $x = 1$).

4.10)

Theorem 2 *If* x *is a real number, then* $|x-2| \leq 3$ *implies that* $-1 \leq x \leq 5$.

Proof. Suppose x is a real number such that $|x-2| \leq 3$. By the definition of absolute value, this means that either

$$x - 2 \geq 0$$

and

$$x - 2 \leq 3$$

or that

$$x - 2 < 0$$

and

$$2 - x \leq 3.$$

In the first case, we have $x \geq 2$ and $x \leq 5$, i.e., $2 \leq x \leq 5$, which implies also that $-1 \leq x \leq 5$. In the second case, we have $x < 2$ and $x \geq -1$, i.e., $-1 \leq x < 2$, which also implies that $-1 \leq x \leq 5$. ■

Proof. ■

4.12) Consider the theorem: "If $xy = 0$ then $x = 0$ or $y = 0$."

a) "Suppose $xy = 0$ and $x \neq 0$. Then dividing both sides of the equation by x we have $y = 0$. Thus if $xy = 0$, then $x = 0$ or $y = 0$." This is a correct proof. It uses the equivalence 3.12(p) from the book.

b) "There are two cases to consider. First suppose that $x = 0$. Then $xy = x(0) = 0$. Similarly, suppose that $y = 0$. Then $xy = x(0) = 0$. In either case, $xy = 0$. Thus if $xy = 0$ then $x = 0$ or $y = 0$." This is not a correct proof. This is an attempt to prove $x = 0$ or $y = 0$ implies $xy = 0$, but that is not the statement. (Plus, there is a typo in the third sentence, which should read "$xy = (0)\,y = 0$.")

4.15)

Theorem 3 *If x is rational and y is irrational, then xy is irrational.*

a) The theorem is false since if $x = 0$ and $y = \sqrt{2}$, then $xy = 0$.
b) The proof given is:
Proof. Suppose x is rational and y is irrational. If xy is rational, then we have $x = p/q$ and $xy = m/n$ for some integers p, q, m, n, with $q \neq 0$ and $n \neq 0$. It follows that

$$y = \frac{xy}{x} = \frac{m/n}{p/q} = \frac{mp}{np}.$$

This implies that y is rational, a contradiction. We conclude that xy must be rational. ■
The problem with the proof is that if $x = 0$, then $y \neq xy/x$.
c) If $x \neq 0$ in addition, then the conclusion is true.

4.19)

Proposition 4 *The sum of any five consecutive integers is divisible by five.*

Proof. Consider five consecutive integers starting with n. Their sum is

$$n + (n+1) + (n+2) + (n+3) + (n+4) = 5\,(n+2)\,,$$

which is divisible by 5. ■

Additional solutions:
4.13a)

Proposition 5 *If x is rational and y is irrational, then $x + y$ is irrational.*

Proof. We will prove that if x is rational, then if $x + y$ is rational then y is rational. (Note: this uses the equivalences $[(p \wedge q) \Rightarrow r] \Leftrightarrow [p \Rightarrow (q \Rightarrow r)] \Leftrightarrow [p \Rightarrow (\sim r \Rightarrow \sim q)]$.) Since x is rational, we can find integers p, q, with $q \neq 0$, such that $x = p/q$ and if $x + y$ is rational, then we also have integers p' and $q' \neq 0$ such that $x + y = p'/q'$. We then find that

$$\begin{aligned} y &= (x+y) - x \\ &= \frac{p'}{q'} - \frac{p}{q} \\ &= \frac{p'q - pq'}{qq'}. \end{aligned}$$

Thus y is rational. ■