Math 323: Homework 5 Solutions

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5.4) Let $A = \{2, 4, 6, 8\}$, $B = \{3, 4, 5, 6\}$, and $C = \{5, 6, 7\}$. Then
f) $(B \cup C) \setminus A = \{3, 5, 7\}$

5.17) Which of the following enable me to conclude that $x \notin A \setminus B$? Note that it is equivalent that $x \notin A$ or $x \in B$.
a) $x \notin A \cup B$. Yes, for then $x \notin A$.

b) $x \in B \setminus A$. Yes, since $x \in B$.

c) $x \in A \cap B$. Yes, since then $x \in B$.

d) $x \in A \cup B$ and $x \notin A$. Yes, since the second statement in the conjunction is sufficient.

e) $x \in A \cup B$ and $x \notin A \cap B$. No, since it may be that $x \in A$ and $x \notin B$. For instance, let $A = \{1, 2\}$, $B = \{2, 3\}$, and $x = 1$.

5.19)

Proposition 1 *If $U = A \cup B$ and $A \cap B = \varnothing$, then $A = U \setminus B$.*

Proof. Suppose we have A and B as stated. Then we must show that $A \subseteq U \setminus B$ and $U \setminus B \subseteq A$. First, suppose that $x \in A$. Since $A \cap B = \varnothing$, $x \notin B$, so $x \in U \setminus B$. Now suppose that $y \in U \setminus B$. Then $y \notin B$. Since $U = A \cup B$ and $y \in U$, we must have that $y \in A$. ■

5.25a)

Proposition 2 $\displaystyle\bigcup_{n \in \mathbb{N}} \left[1, 1 + \frac{1}{n}\right] = [1, 2]$.

Proof. We first show that $\displaystyle\bigcup_{n \in \mathbb{N}} \left[1, 1 + \frac{1}{n}\right] \subseteq [1, 2]$. If $x \in \displaystyle\bigcup_{n \in \mathbb{N}} \left[1, 1 + \frac{1}{n}\right]$, then there exists an integer $n \geq 1$ such that $x \in \left[1, 1 + \frac{1}{n}\right]$. Since $1 + \frac{1}{n} \leq 2$, we see that $x \in [1, 2]$.

Conversely, if $x \in [1, 2]$, then $x \in \left[1, 1 + \frac{1}{n}\right]$ for $n = 1$, thus $x \in \displaystyle\bigcup_{n \in \mathbb{N}} \left[1, 1 + \frac{1}{n}\right]$. ■

Proposition 3 $\displaystyle\bigcap_{n \in \mathbb{N}} \left[1, 1 + \frac{1}{n}\right] = \{1\}$.

Proof. Notice that $1 \in \left[1, 1 + \frac{1}{n}\right]$ for any $n \geq 1$, so $\{1\} \subseteq \displaystyle\bigcap_{n \in \mathbb{N}} \left[1, 1 + \frac{1}{n}\right]$. Now let $x \neq 1$ be a real number. If $x < 1$ or $x > 2$ then clearly $x \notin [1, 2]$. If $1 < x \leq 2$, then there exists a number y such that $1 < y < x$. If we take n to be any integer such that $n > \frac{1}{y-1}$, then $1 + \frac{1}{n} < y < x$, and so $x \notin \left[1, 1 + \frac{1}{n}\right]$. Thus $x \notin \displaystyle\bigcap_{n \in \mathbb{N}} \left[1, 1 + \frac{1}{n}\right]$. ■

5.25c) Let $\mathcal{B} = \{[2, x] : x \in \mathbb{R} \text{ and } x > 2\}$.

Proposition 4 $\bigcup_{B\in\mathcal{B}} B = [2,\infty)$.

Proof. If $x \in \bigcup_{B\in\mathcal{B}} B$, then there exists $y > 2$ such that $x \in [2,y] \subseteq [2,\infty)$. Conversely, suppose $x > 2$. Then $x \in [2,x]$, so $x \in \bigcup_{B\in\mathcal{B}} B$. ■

Proposition 5 $\bigcap_{B\in\mathcal{B}} B = \{2\}$.

Proof. Certainly $2 \in [2,y]$ for every $y > 2$. Conversely, suppose $x \neq 2$. If $x < 2$, then $x \notin [2,3]$. If $x > 2$, then $x > x - \frac{(x-2)}{2} > 2$, and so $x \notin \left[2, x - \frac{(x-2)}{2}\right]$. Thus $x \notin \bigcap_{B\in\mathcal{B}} B$. ■

Bonus solutions:
5.4) Let $A = \{2,4,6,8\}$, $B = \{3,4,5,6\}$, and $C = \{5,6,7\}$. Then

a) $A \cap B = \{4,6\}$

b) $A \cup B = \{2,3,4,5,6,8\}$

c) $A \setminus B = \{2,8\}$

d) $B \cap C = \{5,6\}$

e) $B \setminus C = \{3,4\}$

f) $(B \cup C) \setminus A = \{3,5,7\}$

g) $(A \cap C) \setminus B = \varnothing$

h) $C \setminus (A \cup B) = \{7\}$

5.25b) Let $\mathcal{B} = \left\{\left(1, 1 + \frac{1}{n}\right) : n \in \mathbb{N}\right\}$.

Proposition 6 $\bigcup_{B\in\mathcal{B}} B = (1,2)$.

Proof. If $x \in (1,2)$, then since $(1,2) \in \mathcal{B}$, x is in the union. Thus, $(1,2) \subseteq \bigcup_{B\in\mathcal{B}} B$.

Conversely, suppose $x \in \left(1, 1 + \frac{1}{n}\right)$ for some n. Then since $1 + \frac{1}{n} \leq 2$, $x \in (1,2)$. Thus $\bigcup_{B\in\mathcal{B}} B \subseteq (1,2)$. ■

Proposition 7 $\bigcap_{B\in\mathcal{B}} B = \varnothing$.

Proof. We just need to show that there are no real numbers that are in $\left(1, 1 + \frac{1}{n}\right)$ for every $n \in \mathbb{N}$. This is clear if $x \leq 1$ or $x \geq 2$. Otherwise, $x \in (1,2)$. In this case, there exists n such that $1 + \frac{1}{n} < x$. (This can be shown by taking n an integer larger than $1/(x-1)$.) It follows that x is not in $\left(1, 1 + \frac{1}{n}\right)$ and hence not in the intersection. ■

5.26a)

Proposition 8 $B \cup \left[\bigcap_{j\in J} A_j\right] = \bigcap_{j\in J} (B \cup A_j)$

Proof. Suppose $x \in B \cup \left[\bigcap_{j\in J} A_j\right]$. Then $x \in B$ or for all $j \in J$, $x \in A_j$. If $x \in B$, then $x \in B \cup A_j$ for all j. If $x \in A_j$ for all $j \in J$, then $x \in B \cup A_j$ for all $j \in J$. Hence $x \in \bigcap_{j\in J} (B \cup A_j)$.

Conversely, suppose $x \in \bigcap_{j\in J} (B \cup A_j)$. Then for all $j \in J$, $x \in B$ or $x \in A_j$. Thus either $x \in B$ or $x \in A_j$ for all j. Hence $x \in B \cup \left[\bigcap_{j\in J} A_j\right]$ ■