Math 323: Homework 6 Solutions

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6.11a) Let R be the relation on $\mathbb{N}$ given by xRy iff x divides y.

Proposition 1 *R is reflexive and transitive, but not symmetric.*

Proof. We check the properties individually:

- Reflexive: If $n \in \mathbb{N}$, then $n = (1)\,n$, so n divides n. The relation is reflexive.
- Symmetric: This property is not satisfied. Notice that 2 divides 4 but 4 does not divide 2, so $2R4$ but $4 \not R 2$.
- Transitive: Let $x, y, z \in \mathbb{N}$ such that xRy and yRz, so there are integers m, n such that $y = mx$ and $z = ny$. Thus there $z = (mn)\,x$, so x divides z and xRz. Thus the relation is transitive.

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6.11b) Let X be a set and let R be the relation "$\subseteq$" defined on subsets of X.

Proposition 2 *R is reflexive and transitive, but not symmetric unless $X = \varnothing$.*

Proof. We check the properties individually:

- Reflexive: If $A \subseteq X$, then $A \subseteq A$, so ARA. Thus R is reflexive.
- Symmetric: This property is not satisfied unless X is the empty set, since we see that $\varnothing \subseteq X$, but $X \nsubseteq \varnothing$, so $\varnothing RX$ and $X \not R \varnothing$. If X is the empty set, then it is clearly true.
- Transitive: Let A, B, C be subsets of X. Suppose ARB and BRC. Then $A \subseteq B$ and $B \subseteq C$. It follows that $A \subseteq C$ since if $x \in A$, then $x \in B$ then $x \in C$. Thus ARC. Thus the relation is transitive.

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6.20) Let R be a relation on $\mathbb{Z}$ defined by xRy iff $x - y = 3k$ for some integer k.

Proposition 3 *R is an equivalence relation.*

Proof. We check the properties of an equivalence relation individually:

- Reflexive: Let $x \in \mathbb{Z}$. Then $x - x = 0 = 3\,(0)\,,$ so xRx.
- Symmetric: Let $x, y \in \mathbb{Z}$ such that xRy. Thus there exists $k \in \mathbb{Z}$ such that $x - y = 3k$. It follows that $y - x = 3\,(-k)\,,$ so yRx.
- Transitive: Let $x, y, z \in \mathbb{Z}$ such that xRy and yRz. Thus there exist k and k' in $\mathbb{Z}$ such that $x - y = 3k$ and $y - z = 3k'$. We then see that

$$\begin{aligned} x - z &= x - y + y - z \\ &= 3k + 3k' \\ &= 3\,(k + k')\,. \end{aligned}$$

  Thus xRz.

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The equivalence class $[5] = E_5$ consists of all integers equivalent to 5, i.e., all $x \in \mathbb{Z}$ such that $5 - x = 3k$ for some integer k. Thus

$$\begin{aligned}[5] &= \{5 - 3k : k \in \mathbb{Z}\} \\ &= \{2 + 3k : k \in \mathbb{Z}\}.\end{aligned}$$

We note that there are three equivalence classes: $[0], [1], [2]$, since we can always see that, for $x \in \mathbb{Z}$,

$$[x] = \left[x - 3\left(\left\lfloor \frac{x}{3} \right\rfloor\right)\right].$$

($\lfloor x \rfloor$, "the floor of x," denotes the largest integer less than or equal to x.) Notice that

$$\frac{x}{3} - 1 < \left\lfloor \frac{x}{3} \right\rfloor \leq \frac{x}{3},$$

so

$$-x \leq -3\left\lfloor \frac{x}{3} \right\rfloor < 3 - x,$$

and

$$0 \leq x - 3\left(\left\lfloor \frac{x}{3} \right\rfloor\right) < 3.$$

Thus $[x]$ always equals $[0]$ or $[1]$ or $[2]$.

6.25) Define a relation R on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ by $(a, b) R (x, y)$ iff $ay = bx$.

Proposition 4 *R is an equivalence relation.*

Proof. We prove each property individually:

- Reflexive: For any $(a, b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, we have that $ab = ba$, so $(a, b) R (a, b)$.
- Symmetric: Suppose $(a, b) R (x, y)$. Then $ay = bx$, so $xb = ya$ and $(x, y) R (a, b)$.
- Transitive: Suppose $(a, b) R (c, d)$ and $(c, d) R (e, f)$. Then $ad = bc$ and $cf = de$. Thus, since $d \neq 0$,

  $$af = \frac{adf}{d} = \frac{bcf}{d} = \frac{bde}{d} = be.$$

  Thus, $(a, b) R (e, f)$.

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The equivalence classes are in correspondence with rational numbers in the following sense. The ordered pair $(a, b) R (x, y)$ if and only if $\frac{a}{b} = \frac{x}{y}$, i.e., they represent the same rational number. Thus each equivalence class consists of all possible representations of rational numbers as a quotient of integers, and the set of all equivalence classes corresponds to the set of all rational numbers.