Math 323: Homework 7 Solutions

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7.4c) Consider all possible functions $f : \{1,2\} \to \{5,6\}$. There are four possible functions: $f_1 = \{(1,5),(2,5)\}$, $f_2 = \{(1,5),(2,6)\}$, $f_3 = \{(1,6),(2,5)\}$, $f_4 = \{(1,6),(2,6)\}$.

7.6b) Let $A \subseteq \mathbb{R}$ and define $f : A \to \mathbb{R}$ as $f(x) = |2x-1|$. This function is injective on sets $(-\infty, \frac{1}{2}]$ or $[\frac{1}{2}, \infty)$ or other interesting sets like $(-\infty, a] \cup [\frac{1}{2}, 1-a)$ and $(-\infty, a) \cup [\frac{1}{2}, 1-a]$ for $a < \frac{1}{2}$.

7.8a) Let S be the set of all circles in the plane. Define $f : S \to [0,\infty)$ by $f(C) =$ area of C, for all $C \in S$. The function f is clearly not injective, since one can consider the circle of radius 1 centered at the origin and the circle of radius 1 centered at $(0,1)$ both have the same area. It is not surjective, because circles cannot have zero radius (unless you consider points to be circles of zero radius, in which case it is surjective.

7.29a) Let $f : A \to B$ and suppose there exists a function $g : B \to A$ such that $g \circ f = \iota_A$ and $f \circ g = \iota_B$. Then f is bijective.
Proof. We first show that f is injective. Suppose $f(x) = f(y)$. Then,

$$y = (g \circ f)(y) = g(f(y)) = g(f(x)) = g \circ f(x) = x.$$

Now we show f is surjective. Let $x \in B$. Then let $y = g(x)$. We see that

$$f(y) = f(g(x)) = x.$$

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b) $g = f^{-1}$.
Proof. To show two functions are the same, we need to prove that for all $b \in B$, $g(b) = f^{-1}(b)$.
Let $b \in B$. Then $a = f^{-1}(b)$ iff $f(a) = b$. Then we must have that $a = g(f(a)) = g(b)$. Since $a = f^{-1}(b)$, we have showns that $f^{-1}(b) = g(b)$. ■

7.33) Suppose that $f : A \to B$ is a function. Define a relation R on A by xRy iff $f(x) = f(y)$.

Proposition 1 *R is an equivalence relation on A.*

Proof. We check the properties individually:

1. Reflexive: Let $x \in A$. Then $f(x) = f(x)$, so xRx.
2. Symmetric: Let $x, y \in A$ such that xRy, i.e., $f(x) = f(y)$. Then $f(y) = f(x)$ so yRx.
3. Transitive. Suppose $x, y, z \in A$ such that xRy and yRz, i.e., $f(x) = f(y)$ and $f(y) = f(z)$. Thus $f(x) = f(z)$, so xRz.

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Proposition 2 *For any $x \in A$, let E_x be the equivalence class of x, and let E be the collection of all equivalence classes. The function $g : A \to E$ defined by $g(x) = E_x$ is surjective.*

Proof. Let $F \in E$, then F is the equivalence class of some $x \in A$ (since equivalence classes are not empty), i.e., there exists $x \in A$ such that $F = E_x$. Thus $F = g(x)$ and g is surjective. ∎

More solutions:

7.4a) Consider all possible functions $f : \{1, 2, 3\} \to \{5\}$. Since there is only one element of the codomain, there is only one possible function, which is $\{(1, 5), (2, 5), (3, 5)\}$.

7.4b) Consider all possible functions $f : \{4\} \to \{5, 6\}$. There are two possible functions: $f_1 = \{(4, 5)\}$ and $f_2 = \{(4, 6)\}$.

7.8b) If T is the set of all circles centered at the origin, then $g : T \to [0, \infty)$ is injective (two circles with the area are the same, since area equals πr^2, so the area determines the radius) and it is surjective if one considers circles of zero radius, otherwise not surjective.