

Math 323: Homework 8 Solutions

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7.12a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$. We defined $f + g$ by $(f + g)(x) = f(x) + g(x)$. It is not true that if f and g are bijective, then the sum $f + g$ is bijective. Consider $f(x) = x$ and $g(x) = -x$. Both are clearly bijective (they are their own inverses). But the sum $f + g$ is equal to the constant function zero, which is clearly not bijective.

7.12b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$. We defined fg by $(fg)(x) = f(x)g(x)$. It is not true that if f and g are bijective, then their product fg is bijective. Consider $f(x) = g(x) = x$. Then $fg(x) = x^2$, which is neither injective nor surjective.

7.20) Suppose $f : A \rightarrow B$ and suppose $C \subseteq A$ and $D \subseteq B$.

a) The statement $f(C) \subseteq D$ iff $C \subseteq f^{-1}(D)$ is true. Suppose $f(C) \subseteq D$ and consider $x \in C$. Then $f(x) \in D$, so $x \in f^{-1}(D)$. Conversely, suppose $C \subseteq f^{-1}(D)$, and let $y \in f(C)$. Thus there exists $x \in C$ such that $y = f(x)$. Since $x \in f^{-1}(D)$, we know that $f(x) \in D$, i.e., $y \in D$.

b) If f is injective and $D \subseteq \text{rng}(f)$ (in particular, if f is bijective), then $f(C) = D$ iff $C = f^{-1}(D)$.

Proof. By part (a), $f(C) \subseteq D$ iff $C \subseteq f^{-1}(D)$, thus we need that if f is injective, then $D \subseteq f(C)$ iff $f^{-1}(D) \subseteq C$. First suppose that $D \subseteq f(C)$. Let $x \in f^{-1}(D)$. Then $f(x) \in D$, and hence $f(x) \in f(C)$, i.e., there exists $x' \in C$ such that $f(x') = f(x)$. Since f is injective, $x = x'$, so $x \in C$.

Conversely, suppose $f^{-1}(D) \subseteq C$. Let $y \in D$. Since $D \subseteq \text{rng}(f)$, there exists $x \in A$ such that $f(x) = y$. Since $x \in f^{-1}(D)$, $x \in C$. Thus $y = f(x) \in f(C)$. ■

Extra problem:

Consider the following relation on $[-\pi, \pi]$:

$$x \sim y \text{ iff } x = y \text{ or } x, y \in \{-\pi, \pi\}.$$

a) We first show that this is an equivalence relation. Notice that it is reflexive, since $x = x$. It is symmetric, since if $x \sim y$ then $x = y$ or $x, y \in \{-\pi, \pi\}$, which is the same as if we switch x and y . To prove transitivity, we first assume $x \sim y$ and $y \sim z$. We consider two cases. First suppose that $x \notin \{-\pi, \pi\}$. Then $y = x$, and so $y \notin \{-\pi, \pi\}$. Thus $z = y = x$ and $x \sim z$. Now suppose $x \in \{-\pi, \pi\}$, then we must have $y \in \{-\pi, \pi\}$ and hence $z \in \{-\pi, \pi\}$, thus $x \sim z$.

b) We describe the equivalence classes. There is one equivalence class for every number between $-\pi$ and π and a single equivalence class containing π and $-\pi$. One can think of this as taking the interval $[-\pi, \pi]$ and gluing one end to the other.

c) Show that

$$f([x]) = (\cos x, \sin x)$$

is a well-defined function $A \rightarrow \mathbb{R}^2$, where A is the set of equivalence classes of \sim .

We need to show that if $x \sim y$, then $(\cos x, \sin x) = (\cos y, \sin y)$. Certainly if $x = y$ this is true. Now suppose $x = \pi$ and $y = -\pi$. We see that $(\cos \pi, \sin \pi) = (-1, 0) = (\cos(-\pi), \sin(-\pi))$. Since \sim is symmetric, this is sufficient.

d) Show that if we set the codomain to be

$$B = \{(\cos t, \sin t) \in \mathbb{R}^2 : t \in \mathbb{R}\},$$

then $f : A \rightarrow B$ is a bijection. Hint: you can use inverse trig functions, but be careful of where they exist and what their domains and ranges are!

First we show that f is injective. Suppose $f([x]) = f([y])$, so $\cos x = \cos y$ and $\sin x = \sin y$. For $x \in [-\pi, \pi]$, $\cos x = \cos y$ only if $x = \pm y$. If $x = y$, we are done. Suppose $x = -y$. Then we see that $\sin x = \sin y = \sin(-x) = -\sin x$. So we must have $\sin x = 0$, so $x = \pm\pi$. Thus $[x] = [y]$. We now show that f is surjective. Since $\cos t$ and $\sin t$ are periodic with period 2π , for any $t \in \mathbb{R}$, if we let

$$x = t - 2\pi \left\lfloor \frac{t}{2\pi} + \frac{1}{2} \right\rfloor,$$

we have $\sin x = \sin t$, $\cos x = \cos t$, and

$$-\pi = t - 2\pi \left(\frac{t}{2\pi} + \frac{1}{2} \right) \leq t - 2\pi \left\lfloor \frac{t}{2\pi} \right\rfloor < t - 2\pi \left(\frac{t}{2\pi} - \frac{1}{2} \right) = \pi,$$

so $f([x]) = (\cos t, \sin t)$ and f is surjective.

8.3c)

Proposition 1 *The sets $S = [0, 1)$ and $T = (0, 1)$ are equinumerous.*

Proof. We can always insert one element into an infinite set by taking out a countable set and shifting it. We can define the following function $f : T \rightarrow S$ by

$$f(x) = \begin{cases} x & \text{if } x \neq \frac{1}{n+1} \text{ for all } n \in \mathbb{N} \\ \frac{1}{n} & \text{if } x = \frac{1}{n+1} \text{ for some } n \in \mathbb{N} \setminus \{1\} \\ 0 & \text{if } x = \frac{1}{2} \end{cases}$$

One can show it is bijective directly, or see that it has an inverse function $g : S \rightarrow T$ defined by

$$g(y) = \begin{cases} y & \text{if } y \neq \frac{1}{n+1} \text{ for all } n \in \mathbb{N} \text{ and } y \neq 0 \\ \frac{1}{n+2} & \text{if } y = \frac{1}{n+1} \text{ for some } n \in \mathbb{N} \\ \frac{1}{2} & \text{if } y = 0 \end{cases}$$

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8.3e)

Proposition 2 *The sets $S = (0, 1)$ and $T = \mathbb{R}$ are equinumerous.*

Proof. There are several ways to get a bijection. The key idea is that we need a function that takes a finite interval to an infinite interval. These are functions like $\tan x$, $\frac{1}{x}$, and rational numbers. Also, one can compose several bijections together to get a bijection with the appropriate domain. So, for instance, $f(x) = \tan x$ gives a bijection between $(-\frac{\pi}{2}, \frac{\pi}{2})$ and \mathbb{R} , and so one can then scale and shift by precomposing with the function $g(x) = \pi x - \frac{\pi}{2}$ to get the function $\tan(\pi x - \frac{\pi}{2})$. Here are some bijections $S \rightarrow T$:

$$\begin{aligned} f_1(x) &= \tan\left(\pi x - \frac{\pi}{2}\right) \\ f_2(x) &= \frac{x - \frac{1}{2}}{x(1-x)} \\ f_3(x) &= \begin{cases} \frac{1}{x - \frac{1}{4}} & \text{if } x \in (0, \frac{1}{4}) \\ \frac{1}{x - \frac{3}{4}} & \text{if } x \in (\frac{3}{4}, 1) \\ 16x - 8 & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \end{cases} \end{aligned}$$

One can prove these are bijections by considering regions where they are increasing and decreasing, or sometimes by finding inverse functions (for instance, the inverse to f_3 is

$$g_3(y) = \begin{cases} \frac{1}{y} + \frac{1}{4} & \text{if } y < -4 \\ \frac{1}{y} + \frac{3}{4} & \text{if } y > 4 \\ \frac{1}{6}(y + 8) & \text{if } x \in [-4, 4] \end{cases}$$

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More Solutions:

7.30) Suppose $g : A \rightarrow C$ and $h : B \rightarrow C$. If h is bijective, then there exists a function $f : A \rightarrow B$ such that $g = h \circ f$.

Proof. Since h is bijective, there is a function $h^{-1} : C \rightarrow B$. If we define f to be $h^{-1} \circ g$, then

$$h \circ f = h \circ h^{-1} \circ g = \iota_C \circ g = g.$$

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