Math 323: Homework 9 Solutions

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March 14, 2013

7.12a) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. We defined $f + g$ by $(f+g)(x) = f(x) + g(x)$. It is not true that if f and g are bijective, then the sum $f + g$ is bijective. Consider $f(x) = x$ and $g(x) = -x$. Both are clearly bijective (they are their own inverses). But the sum $f + g$ is equal to the constant function zero, which is clearly not bijective.

7.12b) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. We defined fg by $(fg)(x) = f(x)g(x)$. It is not true that if f and g are bijective, then their product fg is bijective. Consider $f(x) = g(x) = x$. Then $fg(x) = x^2$, which is neither injective nor surjective.

7.20) Suppose $f : A \to B$ and suppose $C \subseteq A$ and $D \subseteq B$.

a) The statement $f(C) \subseteq D$ iff $C \subseteq f^{-1}(D)$ is true. Suppose $f(C) \subseteq D$ and consider $x \in C$. Then $f(x) \in D$, so $x \in f^{-1}(D)$. Conversely, suppose $C \subseteq f^{-1}(D)$, and let $y \in f(C)$. Thus there exists $x \in C$ such that $y = f(x)$. Since $x \in f^{-1}(D)$, we know that $f(x) \in D$, i.e., $y \in D$.

b) If f is injective and $D \subseteq rng(f)$ (in particular, if f is bijective), then $f(C) = D$ iff $C = f^{-1}(D)$.

Proof. By part (a), $f(C) \subseteq D$ iff $C \subseteq f^{-1}(D)$, thus we need that if f is injective, then $D \subseteq f(C)$ iff $f^{-1}(D) \subseteq C$. First suppose that $D \subseteq f(C)$. Let $x \in f^{-1}(D)$. Then $f(x) \in D$, and hence $f(x) \in f(C)$, i.e., there exists $x' \in C$ such that $f(x') = f(x)$. Since f is injective, $x = x'$, so $x \in C$.

Conversely, suppose $f^{-1}(D) \subseteq C$. Let $y \in D$. Since $D \subseteq rng(f)$, there exists $x \in A$ such that $f(x) = y$. Since $x \in f^{-1}(D)$, $x \in C$. Thus $y = f(x) \in f(C)$. ∎

7.30) Suppose $g : A \to C$ and $h : B \to C$. If h is bijective, then there exists a function $f : A \to B$ such that $g = h \circ f$.

Proof. Since h is bijective, there is a function $h^{-1} : C \to B$. If we define f to be $h^{-1} \circ g$, then

$$h \circ f = h \circ h^{-1} \circ g = \iota_C \circ g = g.$$

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Extra problem:

Consider the following relation on $[-\pi, \pi]$:

$$x \sim y \text{ iff } x = y \text{ or } x, y \in \{-\pi, \pi\}.$$

a) We first show that this is an equivalence relation. Notice that it is reflexive, since $x = x$. It is symmetric, since if $x \sim y$ then $x = y$ or $x, y \in \{-\pi, \pi\}$, which is the same as if we switch x and y. To prove transitivity, we first assume $x \sim y$ and $y \sim z$. We consider two cases. First suppose that $x \notin \{-\pi, \pi\}$. Then $y = x$, and so $y \notin \{-\pi, \pi\}$. Thus $z = y = x$ and $x \sim z$. Now suppose $x \in \{-\pi, \pi\}$, then we must have $y \in \{-\pi, \pi\}$ and hence $z \in \{-\pi, \pi\}$, thus $x \sim z$.

b) We describe the equivalence classes. There is one equivalence class for every number between $-\pi$ and π and a single equivalence class containing π and $-\pi$. One can think of this as taking the interval $[-\pi, \pi]$ and gluing one end to the other.

c) Show that

$$f([x]) = (\cos x, \sin x)$$

is a well-defined function $A \to \mathbb{R}^2$, where A is the set of equivalence classes of $\sim$.

We need to show that if $x \sim y$, then $(\cos x, \sin x) = (\cos y, \sin y)$. Certainly if $x = y$ this is true. Now suppose $x = \pi$ and $y = -\pi$. We see that $(\cos \pi, \sin \pi) = (-1, 0) = (\cos(-\pi), \sin(-\pi))$. Since $\sim$ is symmetric, this is sufficent.

d) Show that if we set the codomain to be

$$B = \left\{(\cos t, \sin t) \in \mathbb{R}^2 : t \in \mathbb{R}\right\},$$

then $f : A \to B$ is a bijection. Hint: you can use inverse trig functions, but be careful of where they exist and what their domains and ranges are!

First we show that f is injective. Suppose $f([x]) = f([y])$, so $\cos x = \cos y$ and $\sin x = \sin y$. For $x \in [-\pi, \pi]$, $\cos x = \cos y$ only if $x = \pm y$. If $x = y$, we are done. Suppose $x = -y$. Then we see that $\sin x = \sin y = \sin(-x) = -\sin x$. So we must have $\sin x = 0$, so $x = \pm\pi$. Thus $[x] = [y]$. We now show that f is surjective. Since $\cos t$ and $\sin t$ are periodic with period 2π, for any $t \in \mathbb{R}$, if we let

$$x = t - 2\pi \left\lfloor \frac{t}{2\pi} + \frac{1}{2} \right\rfloor,$$

we have $\sin x = \sin t$, $\cos x = \cos t$, and

$$-\pi = t - 2\pi\left(\frac{t}{2\pi} + \frac{1}{2}\right) \leq t - 2\pi \left\lfloor \frac{t}{2\pi} \right\rfloor < t - 2\pi\left(\frac{t}{2\pi} - \frac{1}{2}\right) = \pi,$$

so $f([x]) = (\cos t, \sin t)$ and f is surjective.

8.3c)

Proposition 1 *The sets $S = [0,1)$ and $T = (0,1)$ are equinumerous.*

Proof. We can always insert one element into an infinite set by taking out a countable set and shifting it. We can define the following function $f : T \to S$ by

$$f(x) = \begin{cases} x & \text{if } x \neq \frac{1}{n+1} \text{ for all } n \in \mathbb{N} \\ \frac{1}{n} & \text{if } x = \frac{1}{n+1} \text{ for some } n \in \mathbb{N} \backslash \{1\} \\ 0 & \text{if } x = \frac{1}{2} \end{cases}$$

One can show it is bijective directly, or see that it has an inverse function $g : S \to T$ defined by

$$g(y) = \begin{cases} y & \text{if } y \neq \frac{1}{n+1} \text{ for all } n \in \mathbb{N} \text{ and } y \neq 0 \\ \frac{1}{n+2} & \text{if } y = \frac{1}{n+1} \text{ for some } n \in \mathbb{N} \\ \frac{1}{2} & \text{if } y = 0 \end{cases}$$

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8.3e)

Proposition 2 *The sets $S = (0, 1)$ and $T = \mathbb{R}$ are equinumerous.*

Proof. There are several ways to get a bijection. The key idea is that we need a function that takes a finite interval to an infinite interval. These are functions like $\tan x$, $\frac{1}{x}$, and rational numbers. Also, one can compose several bijections together to get a bijection with the appropriate domain. So, for instance, $f(x) = \tan x$ gives a bijection between $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\mathbb{R}$, and so one can then scale and shift by precomposing with the function $g(x) = \pi x - \frac{\pi}{2}$ to get the function $\tan\left(\pi x - \frac{\pi}{2}\right)$. Here are some bijections $S \to T$:

$$\begin{aligned} f_1(x) &= \tan\left(\pi x - \frac{\pi}{2}\right) \\ f_2(x) &= \frac{x - \frac{1}{2}}{x(1-x)} \\ f_3(x) &= \begin{cases} \frac{1}{x - \frac{1}{4}} & \text{if } x \in \left(0, \frac{1}{4}\right) \\ \frac{1}{x - \frac{3}{4}} & \text{if } x \in \left(\frac{3}{4}, 1\right) \\ 16x - 8 & \text{if } x \in \left[\frac{1}{4}, \frac{3}{4}\right] \end{cases} \end{aligned}$$

One can prove these are bijections by considering regions where they are increasing and decreasing, or sometimes by finding inverse functions (for instance, the inverse to f_3 is

$$g_3(y) = \begin{cases} \frac{1}{y} + \frac{1}{4} & \text{if } y < -4 \\ \frac{1}{y} + \frac{3}{4} & \text{if } y > 4 \\ \frac{1}{6}(y + 8) & \text{if } x \in [-4, 4] \end{cases}$$

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More Solutions:

7.13b) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Defined fg by $(fg)(x) = f(x) g(x)$. It is not true that if f and g are bijective, then fg is bijective. Consider $f(x) = g(x) = x$. Then both functions are bijective, but $(fg)(x) = x^2$, which is not bijective on $\mathbb{R}$.

8.4)

Proposition 3 *Suppose $m < n$. Then $(0, 1)$ and (m, n) are equinumerous.*

Proof. The function $f : (0, 1) \to (m, n)$ defined by $f(x) = m + nx$ is a bijection if $n \neq 0$. If $n = 0$, then we can take $f(x) = n - mx$, and this is a bijection. (A linear function is injective if the slope is not zero. By looking at the image, we see that the function is a bijection. ■

Proposition 4 *Any two open intervals (a_1, b_1) and (a_2, b_2) are equinumerous.*

Proposition 5 *Since we have bijections $f_1 : (0, 1) \to (a_1, b_1)$ and $f_2 : (0, 1) \to (a_2, b_2)$, the map $f_2 \circ f_1^{-1} : (a_1, b_1) \to (a_2, b_2)$ is a bijection (the composition of bijections is a bijection).*

8.5)

Proposition 6 *If $S \setminus T \sim T \setminus S$ then $S \sim T$..*

Proof. Suppose there is a bijection $f : S \setminus T \to T \setminus S$. Since we have that $S = (S \setminus T) \cup (S \cap T)$ (this follows quickly since if $x \in S$, then $x \in S$ and $x \notin T$ or else $x \in S$ and $x \in T$). Thus we can construct the function $g : S \to T$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in S \setminus T \\ x & \text{if } x \in S \cap T \end{cases}$$

Note that this is well-defined since the sets $(S \setminus T)$ and $(S \cap T)$ are disjoint. The function g is surjective since if $x \in T$ then $x \in S \cap T$ or $x \in T \setminus S$. In the first case $g(x) = x$ and in second case, $g\left(f^{-1}(x)\right) = x$. Thus, g is surjective. To prove injectivity, suppose $g(x) = g(x')$. Then if $g(x) \in T \setminus S$, then $x = f^{-1}(g(x)) = f^{-1}(g(x')) = x'$. If $g(x) \in S \cap T$ then $x = g(x) = g(x') = x'$. Thus g is a bijection. ■

8.10) If S is denumerable, then S is equinumerous with a proper subset of itself.

Proof. Since S is denumerable, there exists a bijection $f : \mathbb{N} \to S$. Consider the set $T = S \setminus \{f(1)\}$. Then consider the function $g : \mathbb{N} \to T$ given by $g(n) = f(n+1)$. It is injective, since if $g(m) = g(n)$, then $f(m+1) = f(n+1)$, and since f is injective, $m+1 = n+1$. Thus $m = n$. Furthermore, it is surjective, since for any $x \in T$, $x \in S$, so $x = f(n)$ for some $n > 1$. Thus $x = f(m+1)$ for some $m \in \mathbb{N}$ $(m = n-1)$, i.e., $x = g(m)$. Thus T is denumerable, and equinumerous to S. ■