Math 323: Some induction examples

David Glickenstein

April 4, 2013

10.4)

Proposition 1 $1^3+2^2+3^3+\cdots+n^3=\frac{1}{4}n^2(n+1)^2$ *for all* $n\in\mathbb{N}$.

Proof. We induct on n. Let $P(n)$ be the statement that $1^3+2^2+3^3+\cdots+n^3=\frac{1}{4}n^2(n+1)^2$. $P(1)$ is the statement that $1^3=\frac{1}{4}(1)^2(1+1)^2$, which is easily checked. Now suppose $P(n)$ is true. We compute

$$\begin{aligned} 1^3+2^2+3^3+\cdots+n^3+(n+1)^3 &= \frac{1}{4}n^2(n+1)^2+(n+1)^3 \\ &= \frac{1}{4}\left(n^2+4n+4\right)(n+1)^2 \\ &= \frac{1}{4}(n+1)^2(n+2)^2, \end{aligned}$$

where the first equality uses the inductive hypothesis. This statement is preciecly $P(n+1)$, so the proposition is proven by the principle of mathematical induction. ■

10.12)

Proposition 2 $1+2(2)+3\left(2^2\right)+\cdots+n2^{n-1}=(n-1)2^n+1$ *for all* $n\in\mathbb{N}$.

Proof. We induct on n. Let $P(n)$ be the statement $1+2(2)+3\left(2^2\right)+\cdots+n2^{n-1}=(n-1)2^n+1$. Notice that $P(1)$ is the statement that $1=(1-1)2+1$, which is true. Now suppose $P(n)$. Using the inductive hypothesis $P(n)$, we compute

$$\begin{aligned} 1+2(2)+3\left(2^2\right)+\cdots+n2^{n-1}+(n+1)2^n &= (n-1)2^n+1+(n+1)2^n \\ &= 2n2^n+1=n2^{n+1}+1, \end{aligned}$$

which is the statement $P(n+1)$. By the principle of mathematical induction, the proposition is proven. ■

10.22)

Proposition 3 *If* $1+x>0$ *then* $(1+x)^n\geq 1+nx$ *for all* $n\in\mathbb{N}$.

Proof. For $n=1$, we see that $(1+x)^n=1+x=1+nx$, so the basis for the induction is true. Now suppose the proposition for some n. Consider

$$\begin{aligned} (1+x)^{n+1} &= (1+x)(1+x)^n \\ &\geq (1+x)(1+nx) \\ &= 1+(1+n)x+nx^2 \\ &\geq 1+(1+n)x \end{aligned}$$

where we have used the inductive hypothesis in the second line and the fact that $nx^2\geq 0$ in the last inequality. The proof is completed by the principle of mathematical induction. ■