Questions

1. Consider the set S of real numbers whose decimal expansion $0.d_1d_2d_3\cdots$ terminates after a finite number of decimal places - for example 0.11123, or 0.01928481. Show that S is countable.

2. Show that any finite set S with n elements has exactly 2^n subsets.

Hint: use induction on n. The base case is simple: if a set has only 1 element, it has two subsets: the empty set and the set itself.

3. Let A and B be finite sets, and let $S = \{f : A \to B\}$ be the set of functions from A to B. Show that S is finite.

4. Let A and B be sets; decide if each statement is true or false. If you decide it is true, prove it. If false, give a counter-example.

1. Both A and B finite implies that $A \times B$ is finite
2. A and B uncountable implies that $A \cap B$ uncountable
3. A and B uncountable implies that $A \cup B$ uncountable

5. Prove the following: a polynomial $f(x)$ of degree n which has $n+1$ distinct roots vanishes identically, i.e. $f(x) = 0$ for all $x \in \mathbb{R}$. Hint: use induction, and the fact that if a is a root of the polynomial $f(x)$, we can 'factor it out' by writing $f(x) = (x-a)g(x)$ where $g(x)$ is a polynomial of degree $n-1$.

6. Prove the (symbolic) product rule for the derivative of the product of $n \geq 2$ functions:

$$(f_1f_2\cdots f_n)' = (f_1'f_2f_3\cdots f_n) + (f_1f_2'f_3f_4\cdots f_n) + \ldots + (f_1f_2\cdots f_{n-1}f_n')$$

i.e. the derivative of a product is the derivative of the first times the rest, plus the derivative of the second times the rest, plus the derivative of the third times the rest, and so on, plus finally the derivative of the last times the rest. Hint: use induction on n and the 'usual' product rule, $(fg)' = f'g + g'f$, which you may assume to be true.

7. Let φ and ψ be the roots of the polynomial $x^2 - x - 1$. By the quadratic formula, they are

$$\varphi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \psi = \frac{1-\sqrt{5}}{2}$$

Now consider the Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$ which has the general recursive formula for $n \in \{0, 1, 2, \ldots\}$:

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0,\ F_1 = 1$$

Show that for all $n \geq 0$, we have the identity

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}$$

Note: you will need to use a type of *strong* induction to prove this, namely that you must assume that the formula holds for both F_{n-1} and F_{n-2} in order to prove that it holds for F_n.

8. Prove the following rule, which goes by the name **DeMorgan's Law**: if $n \in \mathbb{N}$ and $S_1, \ldots, S_n$ are sets, then

$$\left(\bigcup_{k=1}^{n} S_k \right)^c = \bigcap_{k=1}^{n} S_k^c$$

Hint: as usual, use induction on n. The base case is trivial (you should still show it is true).

9. Prove that for all $n \in \mathbb{N}$,

$$1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$$