Solutions

1. If we can write S as the countable union of finite sets, we're done because the countable union of countable sets is again countable. To see how to do this, consider

$$S_n = \{x : x = .d_1 d_2 \cdots d_n\}$$

i.e. those numbers whose decimal expansion terminates at the nth place. Claim: S_n is finite for all n. Indeed, S_0 has one element (0), S_1 has 9 elements:

$$S_1 = \{0.1, 0.2, \ldots, 0.9\}$$

and in general S_n has 9^n elements. Then, it should be clear that

$$S = \bigcup_{n \in \mathbb{Z}_{\geq 0}} S_n$$

2. Following the hint, we use induction on n. I like the notation $P(S)$ to denote the collection of subsets of S (the P stands for 'power set').

- Base case: if $|S| = 1$, then S has two subsets: the empty set (which is a subset of any set), and S itself (a set is always a subset of itself).
- Induction hypothesis: If S is a set with $n-1$ elements, then S has 2^{n-1} subsets, i.e. $|P(S)| = 2^{n-1}$.
- Induction step: Let S be a set with n elements and consider an arbitrary element $a \in S$. Now, we write $P(S) = P_1(S) \cup P_2(S)$ where $P_1(S)$ is the collection of subsets of S which contain a and $P_2(S)$ is the collection of subsets of S which do *not* contain a. Also note that $P_1(S) \cap P_2(S) = \emptyset$ since a subset either contains a or it doesn't. Thus $|P(S)| = |P_1(S)| + |P_2(S)|$.

  1) To find $|P_2(S)|$, consider the set $S' = S \backslash \{a\}$. Note that $A \in P_2(S)$ *if and only if* $A \in P(S')$, and hence $|P_2(S)| = |P(S')|$. We know that $|S'| = n-1$, and so $|P(S')| = 2^{n-1}$ by the induction hypothesis.

  2) To get a subset $A \subset S$ such that $a \in A$, we take a subset $A' \subset S'$ and add a: $A = A' \cup \{a\}$. There are 2^{n-1} subsets of S', so there will be exactly 2^{n-1} subsets of S containing a. Hence $|P_1(S)| = 2^{n-1}$.

  3) Finally, we conclude that $|P(S)| = |P_1(S)| + |P_2(S)| = 2^{n-2} + 2^{n-1} = 2 \cdot 2^{n-1} = 2^n$.

3. Recall that, formally, a function is a subset of the Cartesian product of the domain and co-domain. Thus $S = \{f : A \to B\}$ is contained in the collection of *all* subsets of $A \times B$, that is $S \subset P(A \times B)$. Since A and B are finite sets, $|A \times B|$ is finite. As we have just shown in question 2, a finite set has a finite collection of subsets, so $|P(A \times B)|$ is finite. Since every subset of a finite set is finite, S must be finite as well.

4.

1. True. Proof: the statement is obvious if you write $A \times B$ in a (finite) array; there are nm elements, so you can define any bijection you like from $\{1, \ldots, nm\}$ to $A \times B$. To see an explicit bijection, let $\phi_1 : \{1, \ldots, n\} \to A$ and $\phi_2 : \{1, \ldots, m\} \to B$ be bijections. Then, define the function $\phi : \{1, \ldots, n \cdot m\} \to A \times B$ via the following: $\phi(k) = (\phi_1(k_1), \phi_2(k_2))$ where

   $$k_1 = \left\lceil \frac{k}{m} \right\rceil \quad k_2 = k - (k_1 - 1)m$$

   note: $\lceil x \rceil$ is defined to be the smallest integer n such that $n \geq x$. For instance, $\lceil 1/3 \rceil = 1$, $\lceil 4/3 \rceil = 2$, etc. This function will look like this:

   $$\begin{aligned}
   \phi(1) &= (\phi_1(1), \phi_2(1)) \\
   \phi(2) &= (\phi_1(1), \phi_2(2)) \\
   &\ldots \\
   \phi(m) &= (\phi_1(1), \phi_2(m)) \\
   \phi(m+1) &= (\phi_1(2), \phi_2(1)) \\
   \phi(m+2) &= (\phi_1(2), \phi_2(2)) \\
   &\ldots \\
   \phi(2m) &= (\phi_1(2), \phi_2(m) \\
   &\ldots \\
   \phi(nm) &= (\phi_1(n), \phi_2(m))
   \end{aligned}$$

   It should be clear that this is a bijection, but you should prove it to make sure.

2. False. Take $A = [0, 1]$ and $B = [1, 2]$. Then, both A and B are uncountable but $A \cap B = \{1\}$, which is finite.

3. True. Consider the contrapositive: suppose $A \cup B$ is countable. Then, there is a surjective function $f : \mathbb{N} \to A \cup B$. Let $S = f^{-1}(A)$; $S \subset \mathbb{N}$ and so S is countable, thus $f|_S$ is a surjection from a countable set to A, implying that A is countable. Similarly we can show that B must be countable. Hence by the contrapositive, if A and B are uncountable so to is $A \cup B$.

5. As suggested, we proceed by induction:

- Base case: $n = 0$. If the degree of $f(x)$ is zero, then f is a constant function. Thus if $f(x) = 0$ for some x, $f(x) = 0$ for all x.
- Induction hypothesis: If $f(x)$ is a polynomial of degree $n - 1$ which has n distinct roots, then $f(x)$ is identically zero.

- Induction step: Suppose $f(x)$ has degree n and $n+1$ distinct roots. Then, we can factor out one of our roots i.e. we can write $f(x) = (x-a)g(x)$ where $g(x)$ is a polynomial of degree $n-1$ which has n distinct roots. By the induction hypothesis, $g(x)$ is identically zero, and hence $f(x) = (x-a)g(x) = (x-a)0 = 0$ for all x.

6. Again, we proceed by induction.

- Base case: $n = 2$. This is the usual product rule, $(f_1 f_2)' = f_1' f_2 + f_2' f_1$.
- Induction hypothesis: the product rule holds as stated for $n-1 \geq 2$ functions.
- Induction step:

$$(\underbrace{f_1 f_2 \cdots f_{n-1}}_{g} f_n)' = (g f_n)' = g' f_n + f_n' g = (f_1 f_2 \cdots f_{n-1})' f_n + f_n'(f_1 \cdots f_{n-1}) \qquad (\star)$$

We can now rearrange the last term and apply the induction hypothesis to g' to arrive at:

$$\begin{aligned}(\star) &= \left[(f_1' f_2 f_3 \cdots f_{n-1}) + (f_1 f_2' f_3 f_4 \cdots f_{n-1}) + \ldots + (f_1 f_2 \cdots f_{n-1}')\right] f_n + (f_1 \cdots f_{n-1} f_n') \\ &= (f_1' f_2 f_3 \cdots f_n) + (f_1 f_2' f_3 f_4 \cdots f_n) + \ldots + (f_1 f_2 \cdots f_{n-1} f_n')\end{aligned}$$

as desired.

7. We will apply induction.

- Base case: $n = 0$. This is trivial since $\varphi^0 - \psi^0 = 1 - 1 = 0 = F_0$.
- Induction hypothesis: the formula holds for F_{n-1} and F_{n-2}.
- Induction step: we use the recursive formula $F_n = F_{n-1} + F_{n-2}$, apply induction to F_{n-1} and F_{n-2}, and do a bit of algebra:

$$\begin{aligned}F_n = F_{n-1} + F_{n-2} = \frac{\varphi^{n-1} - \psi^{n-1}}{\varphi - \psi} + \frac{\varphi^{n-2} - \psi^{n-2}}{\varphi - \psi} &= \frac{\varphi^{n-1} + \varphi^{n-2} - (\psi^{n-1} + \psi^{n-2})}{\varphi - \psi} \\ &= \frac{\varphi^n(\overbrace{\varphi^{-1} + \varphi^{-2}}^{1}) - \psi^n(\overbrace{\psi^{-1} + \psi^{-2}}^{1})}{\varphi - \psi} \\ &= \frac{\varphi^n - \psi^n}{\varphi - \psi}\end{aligned}$$

You can easily verify that $\varphi^{-1} + \varphi^{-2} = \psi^{-1} + \psi^{-2} = 1$.

8. Induction on n:

- Base case: $n = 1$. If we have just 1 set, $(\cup_{k=1}^{n} S_k)^c = S^c = \cap_{k=1}^{1} S^c$.
- Comment: the $n = 2$ case is proved in the book, and is needed later.
- Induction hypothesis: If $n \in \mathbb{N}$ and $S_1, \ldots, S_n$ are sets, DeMorgan's law holds.
- Suppose we have $n + 1$ sets $S_1, \ldots, S_{n+1}$. Then,

$$\begin{aligned}
\left(\bigcup_{k=1}^{n+1} S_k\right)^c &= \left(\underbrace{\bigcup_{k=1}^{n} S_k}_{A} \cup S_{n+1}\right)^c = A^c \cap S_{n+1}^c \\
\text{Induction Hypothesis} \quad &= \bigcap_{k=1}^{n} S_k^c \cap S_{n+1}^c \\
&= \bigcap_{k=1}^{n+1} S_k^c
\end{aligned}$$

9. Induction on n:

- Base case: $1 = 1^2$.
- Induction hypothesis: $1 + 3 + 5 + \ldots + (2(n-1) - 1) = (n-1)^2$
- Induction step: simply apply the induction hypothesis to the sum up to $(2(n-1)-1)$:

  $1 + 3 + \ldots + (2(n-1) - 1) + 2n - 1 = (n-1)^2 + 2n - 1 = n^2 - 2n + 1 + 2n - 1 = n^2$