MATH 323: FINAL REVIEW

1. Give the negation of the following statements.
  a. $x < 1$ or $y \geq 3$.
  b. If $x \in A$ and $y \in B$, then $xy \notin C$.
  c. For all $w \in A$ there exists $q \in Q$ such that $q < 5w$.

2. Prove or disprove: If n is an integer and m is an even integer, then $n + m$ is even if and only if n is even.

3. Prove or disprove: If x is a real number then $x^2 = x$ if and only if $x = 1$ or $x = 0$.

4. Let $f(x) = x^2$ be a function on the real numbers. Prove or disprove the following: a) For all y there exists x such that $f(x) = y$. b) For all x and y, $f(x) = f(y)$ implies that $x = y$.

5. Consider the following statement: If $a < b + \varepsilon$ for every $\varepsilon > 0$, then $a \leq b$.
  a. State the contrapositive of this statement.
  b. Using the contrapositive, prove the statement is true. (Hint: consider the number $a - b$.)

6. Consider the functions $f_1(x) = x^2$ and $f_2(x) = x^3$. For each of the following, show which of f_1 and f_2 satisfy the condition and which do not (it could be that both or neither satisfy it). Justify your answer.
  a. For all y there exists an x such that $f(x) \geq y$.
  b. There exists y such that for all x, $f(x) \geq y$.
  c. For all x there exists a y such that $f(x) \geq y$.
  d. There exists x such that for all y, $f(x) \geq y$.

7. Consider the statement "Every differentiable function is continuous."
  a. If we think this statement is false, explain how one might go about trying to prove that it is false (simply give an idea of how to start and what would need to be accomplished).
  b. To prove this statement is true, explain how one would go about a direct proof of the statement.
  c. Explain how one would go about proving the statement using its contrapositive.

8. Recall that an integer x is even if there exists an integer m such that $x = 2m$. Prove that for any integer y, y is even iff y^2 is even. (If you wish, you may use the fact that an integer z is not even iff it is odd, i.e., there exists an integer n such that $z = 2n + 1$.)

9. Show that

$$\bigcap_{n \in \mathbb{Z}} (0, e^n) = \varnothing.$$

10. Show that the relation R on $\mathbb{R}$ given by

$$xRy \text{ iff there exists } z \in \mathbb{R} \text{ such that } z \neq 0 \text{ and } x = zy$$

is an equivalence relation. Describe the equivalence classes.

11. Show that the function

$$f : \mathbb{R} \to \mathbb{R}$$

defined by

$$f(x) = \begin{cases} \frac{1}{x-3} & \text{if } x \neq 3 \\ 0 & \text{if } x = 3 \end{cases}$$

is a bijection.

12.

a. Consider the relation on $\mathbb{R}$ given by xRy iff there exists $k \in \mathbb{Z}$ such that $x - y = k$. Show that R is an equivalence relation.

b. Let E be the set of equivalence classes of the relation R given above. Show that $f : E \to \mathbb{R}$ given by $f([x]) = \sin(2\pi x)$ is a well-defined function.

13. Suppose $f : A \to B$ is a function and that S and T are subsets of A and U and V are subsets of B. Prove or give a counterexample to the following:

a. If $S \subseteq T$, then $f(S) \subseteq f(T)$.

b. If $f(S) \subseteq f(T)$, then $S \subseteq T$.

c. If $U \subseteq V$, then $f^{-1}(U) \subseteq f^{-1}(V)$.

14. Suppose $f : A \to B$ and $g : B \to C$ and that $g \circ f$ is bijective. Show that g is surjective and f is injective.

15. Prove or give a counterexample (justify your answer, but don't give long proofs):

a. Every countable set is denumerable.

b. Every subset of a denumerable set is denumerable.

c. The intersection of two countable sets is countable.

16. For the following, state whether the condition is enough to conclude that A and B are equinumerous. Give justification or a counterexample.

a. There exists a function $f : A \to B$ such that $f(A) = B$ and $f^{-1}(B) = A$.

b. A is equinumerous to $\mathbb{N} \times \mathbb{N}$ and B is equinumerous to $\mathbb{Z}$.

c. $A \subseteq B$ and A is infinite and B is countable.

17.

a. Show that $U \cup V \subseteq U \cap V$ if and only if $U = V$.

b. Let $\{B_j : j \in J\}$ be an indexed family of sets. Show that $\bigcup_{i \in J} B_i \subseteq \bigcap_{j \in J} B_j$ iff for all $i, j \in J$, $B_i = B_j$.

18. Prove that

$$1 + 4 + 9 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1).$$

for all $n \in \mathbb{N}$.

19. For these problems, consider the following addition and multiplication on the set $T = \{a, b, c\}$:

$+$	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

$\times$	a	b	c
a	b	a	c
b	a	b	c
c	c	c	c

which make the set into a field, and the order relation given by

$$a < b, \quad b < c, \quad a < c.$$

a. Identify 0 (additive identity), 1 (multiplicative identity), and -1 (additive inverse of multiplicative identity).

b. Show that the order relation does NOT satisfy the axiom O3: For every $x, y, z \in T$, if $y < z$ then $x + y < x + z$.

c. Show that the order relation satisfies the axiom O4: For every element $x > 0$, if $y < z$ then $xy > xz$.

20. Show that

$$|x - y| = 0 \text{ iff } x = y.$$

21. Explain why the following subsets of $\mathbb{R}$ with their corresponding relations are not ordered fields:

a. $[-1, 1]$ with $<$.

b. $\mathbb{R}$ with $\leq$.

c. $\mathbb{R} \backslash \left\{\frac{1}{2}, -\frac{1}{2}, 2, -2\right\}$ with $<$.

22.

Prove that

$$(2)(6)(10)(14)\cdots(4n-2) = \frac{(2n)!}{n!}$$

for all $n \in \mathbb{N}$.

23. Let A be the set

$$A = \{0, 1, 2, 3\}.$$

Consider the relation R on A given by

$$xRy \text{ iff there exists } k \in \mathbb{Z} \text{ such that } x - y = 3k.$$

The following questions all refer to the relation R on A.

a. Show that R is an equivalence relation.

b. Describe the equivalence classes of R.

c. If E is the set of equivalence classes of R, show that $f([x]) = x(x-3)$ is a well defined function $f : E \to \mathbb{R}$.

24.

a. Find the infimum of the set

$$Q = \left\{4 + \frac{1}{n} : n \in \mathbb{N}\right\}$$

and show it is the infimum.

b. Let $K \subseteq \mathbb{R}$ be a set that has a maximum m. Show that m is the supremum of K.

25. If $A \subseteq B$ and both are bounded above, then $\sup A \leq \sup B$ and $\inf A \geq \inf B$. Give an example where $\sup A = \sup B$ and $\inf A = \inf B$ but $A \neq B$.