Review, with some solutions

1. Give the negation of the following statements.

a. $x < 1$ or $y \geq 3$.

b. If $x \in A$ and $y \in B$, then $xy \notin C$.

c. For all $w \in A$ there exists $q \in Q$ such that $q < 5w$.

2. Prove or disprove: If n is an integer and m is an even integer, then $n + m$ is even if and only if n is even.

Answer: Proof of the statement: Suppose n is an integer and m is an even integer, so there is an integer k such that $m = 2k$. Suppose n is even, and so there is an integer ℓ such that $n = 2\ell$. Then $n + m = 2k + 2\ell = 2(k + \ell)$, so $n + m$ is even. Now suppose that $n + m$ is even, so there exists and integer p such that $n + m = 2p$. Thus $n = 2p - m = 2p - 2k = 2(p - k)$. Thus n is even.

3. Prove or disprove: If x is a real number then $x^2 = x$ if and only if $x = 1$ or $x = 0$.

Answer: Proof of the statement: Let x be a real number. If $x = 0$ or $x = 1$, a simple verification confirms that $x^2 = x$. Conversely, if $x^2 = x$ and $x \neq 0$, then we can divide by x and find that $x = 1$.

4. Let $f(x) = x^2$ be a function on the real numbers. Prove or disprove the following: a) For all y there exists x such that $f(x) = y$. b) For all x and y, $f(x) = f(y)$ implies that $x = y$.

Answer: Counterexample to (a): Let $y = -1$. Then since $x^2 \geq 0$, it is not possible to have $f(x) = x^2 = -1$.

Counterexample to (b): Let $x = 1$ and $y = -1$. Then $f(x) = f(1) = 1 = f(-1) = f(y)$, but $x = 1 \neq -1 = y$.

5. Consider the following statement: If $a < b + \varepsilon$ for every $\varepsilon > 0$, then $a \leq b$.

a. State the contrapositive of this statement.

Answer: If $a > b$ then there exists $\varepsilon > 0$ such that $a \geq b + \varepsilon$.

b. Using the contrapositive, prove the statement is true. (Hint: consider the number $a - b$.)

Answer: Suppose $a > b$. Then $a - b > 0$ and so if we let $\varepsilon = a - b$ then $b + \varepsilon = a$.

6. Consider the functions $f_1(x) = x^2$ and $f_2(x) = x^3$. For each of the following, show which of f_1 and f_2 satisfy the condition and which do not (it could be that both or neither satisfy it). Justify your answer.

a. For all y there exists an x such that $f(x) \geq y$.

Answer: True for both. (Both functions get arbitrarily large.)

b. There exists y such that for all x, $f(x) \geq y$.

Answer: True for f_1, which has a lower bound of 0. False for f_2.

c. For all x there exists a y such that $f(x) \geq y$.

Answer: True for both. Given x, we can always find a number smaller than $f(x)$.

d. There exists x such that for all y, $f(x) \geq y$.

Answer: This is false for all functions.

7. Consider the statement "Every differentiable function is continuous."

a. If we think this statement is false, explain how one might go about trying to prove that it is false (simply give an idea of how to start and what would need to be accomplished).

Answer: We would need to find a differentiable function that is not continuous. This is a counterexample.

b. To prove this statement is true, explain how one would go about a direct proof of the statement.

Answer: We suppose f is a differentiable function. Then we need to prove that f is continuous.

c. Explain how one would go about proving the statement using its contrapositive.

Answer: The contrapositive is that every function that is not continuous is not differentiable, so we would suppose f is a function that is not continuous, and then we would need to show that f is not differentiable.

8. Recall that an integer x is even if there exists an integer m such that $x = 2m$. Prove that for any integer y, y is even iff y^2 is even. (If you wish, you may use the fact that an integer z is not even iff it is odd, i.e., there exists an integer n such that $z = 2n + 1$.)

Answer: First we will show that if y is even, then y^2 is even. Suppose y is even. Then there exists an integer m such that $y = 2m$. Then

$$y^2 = (2m)^2 = 4m^2 = 2\left(2m^2\right).$$

Thus y^2 is even.

We will now show that if y is not even, then y^2 is not even. Suppose y is not even, then it is odd. Thus there exists an integer n such that $y = 2n + 1$. Then

$$y^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 2\left(2n^2 + 2n\right) + 1.$$

Hence y^2 is odd, and thus not even.

9. Show that

$$\bigcap_{n \in \mathbb{Z}} (0, e^n) = \varnothing.$$

Proof. We need to show that for any $x \in \mathbb{R}$, $x \notin \bigcap_{n\in\mathbb{Z}} (0, e^n)$, i.e., there exists $n \in \mathbb{Z}$ such that $x \notin (0, e^n)$. Certainly, if $x \leq 0$, then $x \notin (0, e^n)$ for any n. Now suppose $x > 0$. Then there exists an integer n less than $\ln x$, e.g., $\lfloor \log x \rfloor - 1$. Then $n < \log x$, and so $e^n < x$ (since e^x is an increasing function). Thus $x \notin (0, e^n)$ for this choice of n. □

10. Show that the relation R on $\mathbb{R}$ given by

$$xRy \text{ iff there exists } z \in \mathbb{R} \text{ such that } z \neq 0 \text{ and } x = zy$$

is an equivalence relation. Describe the equivalence classes.

Proof. We show the properties individually:

- Reflexive: For any $x \in \mathbb{R}$ $x = (1)\,x$, and so xRx.
- Symmetric: Suppose $x, y \in \mathbb{R}$ and xRy. Then there exists $z \in \mathbb{R}$ such that $z \neq 0$ and $x = zy$. Since $z \neq 0$, we have $y = \frac{1}{z}x$, and so yRx.
- Transitive: Suppose $x, y, w \in \mathbb{R}$ and xRy and yRw. Then there exist $z, z' \in \mathbb{R} \setminus \{0\}$ such that $x = zy$ and $y = z'w$. Then we see that $x = (zz')\,w$ (with $zz' \neq 0$) so xRw.

We claim there are two equivalence classes, one consisting of all nonzero real numbers and one consisting of zero. Note that if $x, y \in \mathbb{R}$ and $x \neq 0$, then if we let $z = \frac{y}{x}$, then $y = zx$, so yRx and x and y are in the same equivalence class. If $x = 0$, then for any yRx, $y = 0$ since $z\,(0) = 0$ for any real number z. □

11. Show that the function

$$f : \mathbb{R} \to \mathbb{R}$$

defined by

$$f\,(x) = \begin{cases} \frac{1}{x-3} & \text{if } x \neq 3 \\ 0 & \text{if } x = 3 \end{cases}$$

is a bijection.

Proof. We must show that f is both surjective and injective. First we consider surjectivity. Let $y \in \mathbb{R}$. If $y \neq 0$, then we can let $x = 3 + \frac{1}{y}$ and we see that $f\,(x) = y$. If $y = 0$, we see that $f\,(0) = y$. Hence f is surjective. Now we consider injectivity. Suppose $f\,(x) = f\,(y)$. If $x = 3$, then $f\,(x) = 0 = f\,(y)$, and since $\frac{1}{x-3}$ is never equal to zero, we must have $y = 3$. If $x \neq 3$, then $f\,(x) = \frac{1}{x-3} \neq 0$. So we must have $y \neq 3$, and thus $f\,(y) = \frac{1}{y-3}$. We now solve

$$\begin{aligned} \frac{1}{x-3} &= \frac{1}{y-3} \\ y - 3 &= x - 3 \\ y &= x. \end{aligned}$$

□

12.

a. Consider the relation on $\mathbb{R}$ given by xRy iff there exists $k \in \mathbb{Z}$ such that $x - y = k$. Show that R is an equivalence relation.

We prove the properties of an equivalence relation separately:
Reflexive: Let $x \in \mathbb{R}$. Then $x - x = 0 \in \mathbb{Z}$, so xRx.
Symmetric: Suppose xRy, so there exists $k \in \mathbb{Z}$ such that $x - y = k$. Then $y - x = -k \in \mathbb{Z}$, so yRx.
Transitive: Suppose xRy and yRz. Then there exists $k, \ell \in \mathbb{Z}$ such that $x - y = k$ and $y - z = \ell$. Adding these yeilds

$$x - z = z - y + y - z = k + \ell \in \mathbb{Z}.$$

Thus, xRz.
We conclude that R is an equivalence relation.

b. Let E be the set of equivalence classes of the relation R given above. Show that $f : E \to \mathbb{R}$ given by $f([x]) = \sin(2\pi x)$ is a well-defined function.

We need to show that if $[x] = [y]$, then $\sin 2\pi x = \sin 2\pi y$. If $[x] = [y]$, then xRy, so $x - y = k$ for some $k \in \mathbb{Z}$. Thus $x = y + k$. We conclude that

$$\sin 2\pi x = \sin(2\pi(y+k)) = \sin(2\pi y + 2\pi k) = \sin 2\pi y.$$

13. Suppose $f : A \to B$ is a function and that S and T are subsets of A and U and V are subsets of B. Prove or give a counterexample to the following:

a. If $S \subseteq T$, then $f(S) \subseteq f(T)$.

Suppose $S \subseteq T$. Let $x \in f(S)$, so there exists $y \in S$ such that $x = f(y)$. Since $S \subseteq T$, $y \in T$, so $x \in f(T)$.

b. If $f(S) \subseteq f(T)$, then $S \subseteq T$.

This is not true. Let $A = S = \mathbb{R}$ and let $T = \{0\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1$. Then $f(S) = f(T) = \{1\}$, but S is not subset of T.

c. If $U \subseteq V$, then $f^{-1}(U) \subseteq f^{-1}(V)$.

Suppose $U \subseteq V$. Let $x \in f^{-1}(U)$, so $f(x) \in U$. Since $U \subseteq V$, we have that $f(x) \in V$, so $x \in f^{-1}(V)$.

14. Suppose $f : A \to B$ and $g : B \to C$ and that $g \circ f$ is bijective. Show that g is surjective and f is injective.

First we show that g is surjective. Let $x \in C$. Since $g \circ f$ is bijective, there exists $y \in A$ such that $g \circ f(y) = x$. But then $g(f(y)) = x$, g is surjective (given x, $f(y)$ maps to x).

Now we show that f is injective. Suppose $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$, and since $g \circ f$ is injective, $x = y$. Thus f is injective.

15. Prove or give a counterexample (justify your answer, but don't give long proofs):

a. Every countable set is denumerable.

False. Finite sets are countable, so $\{1\}$ is countable, but not denumerable.

b. Every subset of a denumerable set is denumerable.

False. Take $\mathbb{N}$. The set $\{1\}$ is a subset but not denumerable.

c. The intersection of two countable sets is countable.

True. If A and B are countable, then $A \cap B$ is a subset of A, and subsets of countable sets are countable.

16. For the following, state whether the condition is enough to conclude that A and B are equinumerous. Give justification or a counterexample.

a. There exists a function $f : A \to B$ such that $f(A) = B$ and $f^{-1}(B) = A$.

b. A is equinumerous to $\mathbb{N} \times \mathbb{N}$ and B is equinumerous to $\mathbb{Z}$.

c. $A \subseteq B$ and A is infinite and B is countable.

17. (20pts) Let $f : A \to B$ and suppose that $C \subseteq B$ and $D \subseteq B$. Show that

$$f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D).$$

18. Prove that

$$1 + 4 + 9 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1).$$

for all $n \in \mathbb{N}$.

19. For these problems, consider the following addition and multiplication on the set $T = \{a, b, c\}$:

$+$	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

$\times$	a	b	c
a	b	a	c
b	a	b	c
c	c	c	c

which make the set into a field, and the order relation given by

$$a < b, \quad b < c, \quad a < c.$$

a. Identify 0 (additive identity), 1 (multiplicative identity), and -1 (additive inverse of multiplicative identity).

b. Show that the order relation does NOT satisfy the axiom O3: For every $x, y, z \in T$, if $y < z$ then $x + y < x + z$.

c. Show that the order relation satisfies the axiom O4: For every element $x > 0$, if $y < z$ then $xy > xz$.

20. Show that

$$|x - y| = 0 \text{ iff } x = y.$$

Clearly, if $x = y$ then $x - y = 0$ and by the definition of absolute value, $|x - y| = 0$. Now suppose $x \neq y$. Then $x > y$ or $x < y$. If $x > y$, then $x - y > 0$ and thus $|x - y| = x - y > 0$. If $x < y$, then $|x - y| = y - x > 0$.

21. Explain why the following subsets of $\mathbb{R}$ with their corresponding relations are not ordered fields:

a. $[-1, 1]$ with $<$.

Addition is not well-defined, since $1 + 1$ is not in $[-1, 1]$.

b. $\mathbb{R}$ with $\leq$.

This does not satisfy the trichotomy property, since $1 \leq 1$ and $1 \geq 1$ and $1 = 1$.

c. $\mathbb{R} \setminus \left\{\frac{1}{2}, -\frac{1}{2}, 2, -2\right\}$ with $<$.

Addition is not well-defined, since $\frac{1}{4} + \frac{1}{4}$ is not in the set.

22.

Prove that

$$(2)(6)(10)(14)\cdots(4n-2)=\frac{(2n)!}{n!}$$

for all $n \in \mathbb{N}$.

The proof is by induction on n. For the basis, we note that

$$2=\frac{2!}{1!}.$$

Now let $P(n)$ be the statement above for a given n. We see that

$$(2)(6)(10)(14)\cdots(4(n+1)-2)=\frac{(2n)!}{n!}(4n+2)$$

by the inductive hypothesis. Note that

$$\frac{(2(n+1))!}{(n+1)!}=\frac{(2n)!(2n+1)(2n+2)}{n!(n+1)}=\frac{(2n)!(4n+2)}{n!},$$

thus

$$(2)(6)(10)(14)\cdots(4(n+1)-2)=\frac{(2(n+1))!}{(n+1)!}.$$

This proves the inductive step (i.e., $P(n)$ implies $P(n+1)$), and the proof is completed by the principle of mathematical induction.

23. Let A be the set

$$A=\{0,1,2,3\}.$$

Consider the relation R on A given by

$$xRy \text{ iff there exists } k \in \mathbb{Z} \text{ such that } x-y=3k.$$

The following questions all refer to the relation R on A.

a. Show that R is an equivalence relation.

b. Describe the equivalence classes of R.

c. If E is the set of equivalence classes of R, show that $f([x])=x(x-3)$ is a well defined function $f: E \to \mathbb{R}$.

24.

a. Find the infimum of the set

$$Q=\left\{4+\frac{1}{n} : n \in \mathbb{N}\right\}$$

and show it is the infimum.

b. Let $K \subseteq \mathbb{R}$ be a set that has a maximum m. Show that m is the supremum of K.

25. If $A \subseteq B$ and both are bounded above, then $\sup A \leq \sup B$ and $\inf A \geq \inf B$. Give an example where $\sup A = \sup B$ and $\inf A = \inf B$ but $A \neq B$.